What do we know about aliquot sequences? (in honor of Richard K. Guy's 100th birthday)

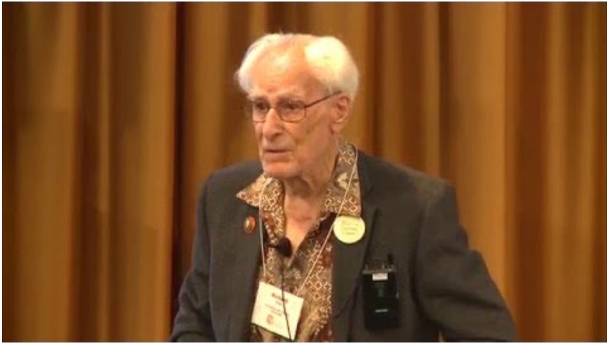
A. S. Mosunov
(joint work with K. Chum, R. K. Guy and M. J. Jacobson, Jr.)

University of Waterloo
Number Theory Seminar

September 29th, 2016

Contents

Richard K. Guy



Richard Guy at CNTA 2016

Part I.
On the Heuristics of Guy and Selfridge

Background

- Let n be a positive integer. Let $s(n)$ denote the sum of the proper divisors of n.
- **Example.** $s(12) = 1 + 2 + 3 + 4 + 6 = 16$.
- Let $s_k(n)$ denote the k-th iterate of s. An *aliquot sequence* starting at n is a sequence of the form

$$n, \; s(n), \; s_2(n) = s(s(n)), \; s_3(n) = s(s(s(n))),$$

and so on.

Background

- Let n be a positive integer. Let $s(n)$ denote the sum of the proper divisors of n.
- **Example.** $s(12) = 1 + 2 + 3 + 4 + 6 = 16$.
- Let $s_k(n)$ denote the k-th iterate of s. An *aliquot sequence* starting at n is a sequence of the form

  $$n, \; s(n), \; s_2(n) = s(s(n)), \; s_3(n) = s(s(s(n))),$$

  and so on.

Background

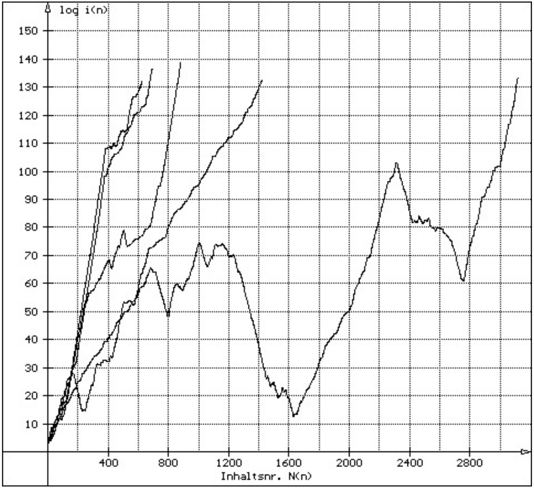
- **Example.** An aliquot sequence starting at 12 is
$$12,\ 16,\ 15,\ 9,\ 4,\ 3,\ 1,\ 0.$$
Thus the sequence terminates.
- **Example.** An aliquot sequence starting at 790 is
$$790,\ 650,\ 652,\ 496,\ 496,\ \ldots.$$
Thus the sequence is eventually periodic with period 1.
- Both are examples of *bounded* aliquot sequences.
- **Catalan-Dickson Conjecture.** Every aliquot sequence is bounded.

On the Heuristics of Guy and Selfridge

- We do not know any n such that the aliquot sequence starting at n is unbounded.
- However, up to 1000 there are 12 possible candidates: $276, 306, 396, 552, 564, 660, 696, 780, 828, 888, 966, 996$.
- The aliquot sequences starting at $276, 552, 564, 660$ and 966 were studied by Derrik Lehmer.

On the Heuristics of Guy and Selfridge

- Lehmer's five, as seen at the top from left to right: 660, 966, 552, 276 and 564.[1]



[1] Data from www.aliquot.de/lehmer.htm.

Conjectures and Heurstics of Guy and Selfridge

- **Guy-Selfridge Counter Conjecture.** There are infinitely many aliquot sequences that are unbounded.
- **Guy-Selfridge Heuristics.** Most of the aliquot sequences starting with even number are unbounded, while most of the aliquot sequences starting with an odd number are bounded.

Part II.
On Guides and Drivers

Guides and Drivers

- In their 1975 paper *What drives an aliquot sequence?* Guy and Selfridge introduced *guides* and *drivers*.
- A *guide* is a number 2^a, together with a subset of the prime factors of $\sigma(2^a)$.
- A *driver* is defined as a number $2^a v$ with $a > 0$, v odd, $v \mid \sigma(2^a)$ and $2^{a-1} \mid \sigma(v)$.
- **Theorem** (Guy and Selfridge, 1975) The only drivers are 2, $2^3 3$, $2^3 3 \cdot 5$, $2^5 3 \cdot 7$, $2^9 3 \cdot 11 \cdot 31$, and the even perfect numbers.

Examples of Driver Dominated Sequences

- $552 = 2^3 3 \cdot 23, \quad s(552) = 2^3 3 \cdot 37$, $s_2(552) = 2^4 3 \cdot 29$, ..., $s_{181}(552) = 2^2 3^2 5 \cdot 7^2 c$.
- $9852 = 2^2 3 \cdot 821, \quad s(9852) = 2^2 3 \cdot 1097$, $s_2(9852) = 2^2 3 \cdot 5 \cdot 293$, ..., $s_{146}(9852) = 2^4 3 \cdot 11 \cdot 31 \cdot c$.
- Despite the tenacity of these drivers, none is expected to live for ever.
- $276 = 2^2 3 \cdot 23$, ..., $s_{169}(276) = 2^2 7^2 p$ with p a prime congruent to 1 mod 4. Then

$$s_{170}(276) = 2 \cdot 5 \cdot 7 \cdot 13 \cdot 829 \cdot 848557 \cdot p.$$

- In order to loose a driver, like in the example above, certain strict conditions have to be satisfied.

Loosing Drivers

- If 2 is a driver of n, then $s(n)$ is odd when n is either a square or twice-a-square.
- The updriver $2 \cdot 3$ can be lost if $n = 2 \cdot d^2 p$, where d is odd and $p = 4k+1$.
- The updriver $2^2 7$ can only get lost if the term is of shape $2^2 7^e d^2 p$ or $2^2 7^e d^2 qr$ where e is even, d is odd, $p = 4k+1$ or $8k+3$, and $q \equiv r \equiv 1 \pmod 4$. By a result of Landau, the total number o numbers less than n with k or less prime factors is

$$\frac{n(\log\log n)^{k-1}}{(k-1)!\log n},$$

so the chances in the above two cases are

$$\frac{1}{8}\frac{2}{\log n}\frac{3}{4} \quad \text{and} \quad \frac{1}{8}\frac{2\log\log n}{\log n}\frac{1}{2^2}.$$

Markov Process

- Using the technique of Devitt (1976), Chum and Jacobson performed a statistical analysis of aliquot sequences.
- **Idea.** One can view an aliquot sequence starting at n as a *Markov process*. Each guide is viewed as a *state*. One records how often aliquot sequences tend to pass from one guide to the other.
- In total, 4000 aliquot sequences got analyzed: eight sets of 500 sequences, with each sequence starting at $2^{16+32r} + 2k$, where $0 \leq r \leq 7$ and $0 \leq k < 500$.
- Out of 4000 sequences, 799 reached a prime, 3179 passed the limit of 2^{288}, and 22 entered a cycle. In total, 2779344 terms got computed.

Data for Each Guide

Guide	Times Seen	Runs	Average Length	Amplification by Term
2	634373	20913	30.3339	-0.438682
$2 \cdot 3$	372308	2478	150.245	0.244404
2^2	655343	64022	10.2362	0.32637
$2^2 \cdot 7$	229949	36446	6.30931	0.0656572
2^3	131710	22518	5.8491	-0.0243489
$2^3 \cdot 3$	102944	5961	17.2696	0.541797
$2^3 \cdot 5$	60520	6662	9.08436	0.3272
$2^3 \cdot 3 \cdot 5$	68080	1592	42.7638	0.808602
2^4	156755	32142	4.87695	0.354399
$2^4 \cdot 31$	128285	1025	125.156	0.412274
2^5	40882	16108	2.53799	0.119586
$2^5 \cdot 3$	31705	5845	5.42429	0.653538
$2^5 \cdot 7$	19529	2384	8.19169	0.356001
$2^5 \cdot 3 \cdot 7$	25753	783	32.8902	0.822831
...				

Part III.
On Geometric Means of k-th Iterates

Previous Results

- In 2003, Bosma and Kane proved that the geometric mean of $s(n)/n$ taken over the first N even integers converges to a constant $\mu \approx 0.9672875 < 1$ when N tends to infinity. The value μ is called the Bosma-Kane constant.
- In 2015, Pomerance proved that the geometric mean of $s_2(n)/s(n)$ taken over the first N even integers excluding 2 converges to the Bosma-Kane constant μ as N tends to infinity.
- Because $\mu < 1$, both results give a strong probabilistic evidence that most of the aliquot sequences starting at an even number are bounded.

Results

- We showed that the geometric means of $s_k(n)/s_{k-1}(n)$ for $n \leq X$ exceed 1 for $X = 2^{37}$ and $k = 6, 7, 8, 9, 10$ when averaged over all even n such that $s_k(n) > 0$. Moreover, as k increases, the geometric means grow, too.
- However, as k remains fixed, the geometric means decrease with the growth of X, possibly approaching the geometric mean of $s(n)/n$.

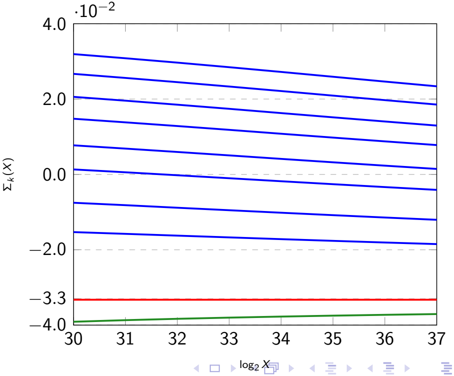
Results

- Let $A_k(X)$ denote the number of even $n \leq X$ such that $s_k(n) > 0$. The graphs display the function

$$\Sigma_k(X) = \frac{1}{A_k(X)} \sum_{\substack{n \leq X \\ 2|n}} \log \frac{s_k(n)}{s_{k-1}(n)}$$

for different values of k as X varies through $2^{30}, 2^{31}, \ldots, 2^{37}$.

- Red line: $k = 1$;
- Green line: $k = 2$;
- Blue lines: from bottom to top correspond to $k = 3, 4, \ldots, 10$.



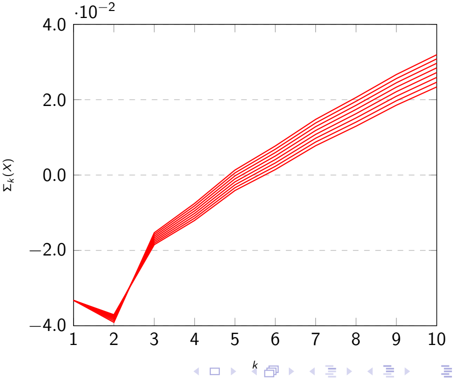
Results

- Let $A_k(X)$ denote the number of even $n \le X$ such that $s_k(n) > 0$. The graphs display the function

$$\Sigma_k(X) = \frac{1}{A_k(X)} \sum_{\substack{n \le X \\ 2|n}} \log \frac{s_k(n)}{s_{k-1}(n)}$$

for different values of X as k varies through $1, 2, \ldots, 10$.

- Red lines: as seen on the right, from top to bottom, correspond to $X = 2^{30}, 2^{31}, \ldots, 2^{37}$.



Pomerance's Conjecture

- The following conjecture was suggested by Carl Pomerance:

Conjecture. *Let k be a positive integer and define $s_0(n) := n$. The geometric mean of $s_k(n)/s_{k-1}(n)$ taken over the first N even integers with $s_k(n) > 0$ converges to the Bosma-Kane constant $\mu \approx 0.9672875$ when N tends to infinity.*

Outline of the Algorithm

1. **Setup.** Suppose we want to iterate through $s_k(n)$ for all even $n \leq X$ and $k = 1, 2, \ldots, K$. Use the algorithm of Moews and Moews to compute $\sigma(n)$ for all $n \leq X$. Store all $\sigma(n)$ into the file `Sigma`.
2. **Tabulating $s(n)$.** Load `Sigma` into memory. Compute $s(n) = \sigma(n) - n$ for each n. If $s(n) \leq X$, store it into the file `Small1`. If $s(n) > X$, store it into the file `Large1`.
3. **Tabulating $s_2(n)$.**
   a) Load `Sigma` into memory.
   b) For each n in `Small1`, compute $s(n) = \sigma(n) - n$ by taking $\sigma(n)$ from `Sigma`.
   c) For each n in `Large1` (in parallel), compute its prime factorization in order to evaluate $s(n) = \sigma(n) - n$.
   d) If $s(n) = 0$, disregard it. If $1 \leq s(n) \leq X$, store it into the file `Small2`. If $s(n) > X$, store it into the file `Large2`.
4. Repeat steps 3a) – 3d) to tabulate $s_3(n)$, $s_4(n)$, and so on.

Tabulating $s_k(n)$ for even $n \leq X = 40$ and $k = 1, 2, 3$

k	Small	Large
0	2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40	
1	1, 3, 6, 7, 8, 16, 10, 15, 21, 22, 14, 36, 16, 28, 31, 20, 22	42, 55, 50
2	1, 6, 1, 7, 15, 8, 9, 11, 14, 10, 15, 28, 1, 22, 14, 17	55, 54, 43
3	6, 1, 9, 7, 4, 1, 10, 8, 9, 28, 14, 10, 1, 17, 1	66

- For our computations, we used Westgrid's supercomputer Hungabee.

Part IV.
On the Tabulation of Untouchable Numbers

Background

- A number n is called *untouchable* if there is no m such that $n = s(m)$. It is called *touchable* otherwise.
- **Pollack-Pomerance Conjecture.** The set of nonaliquot numbers has asymptotic density Δ, where

$$\Delta = \lim_{y \to \infty} \frac{1}{\log y} \sum_{\substack{a \le y \\ 2 | a}} \frac{1}{a} e^{-a/s(a)}.$$

- For $y = 10^{10}$, the summation above yields $\Delta \approx 0.17$.
- Richard Guy suggested that the Bosma-Kane constant μ might be less than one because the geometric mean is taken over *all* even numbers, rather than over *all touchable* even numbers.

Variant of a Goldbach's Conjecture

- **Variant of a Goldbach's Conjecture.** For any odd $n \geq 9$ there exist two distinct odd primes p and q such that $n = 1 + p + q = s(pq)$.
- As a consequence, the number 5 is the only odd untouchable number, since $1 = s(2)$, $3 = s(4)$, $7 = s(8)$, but no such expression exists for 5.
- This variant of a Goldbach's conjecture has been verified computationally by Oliveira e Silva to 4×10^{18}.

Pomerance-Yang Algorithm

The algorithm of Pomerance and Yang allows to tabulate all even touchable/untouchable numbers up to X.

1. Compute $\sigma(n)$ for all odd $n \le X$ such that n is not a perfect square.
2. If $\sigma(n) = n+1$, i.e. n is prime, mark $n+1$ as touchable, since $n+1 = s(n^2)$.
3. Compute $s(2n) = 3\sigma(n) - 2n$, $s(2^{j+1}n) = 2s(2^j n) + \sigma(n)$ for all $j = 1, 2, \ldots$ such that $s(2^j n) \le X$. Mark them all as touchable.
4. For all composite odd $n \le X^{2/3}$, mark every $s(n^2) \le X$ as touchable.

Tabulating Even Untouchable numbers up to $X = 40$

n	$\sigma(n)$	s	n	$\sigma(n)$	s
1			21	32	
3	4	4, 6, 16, 36	23	24	24, 26
5	6	6, 8, 22	25		
7	8	8, 10, 28	27	40	
9		40	29	30	30, 32
11	12	12, 14, 40	31	32	32, 34
13	14	14, 16	33	48	
15	24		35	48	
17	18	18, 20	37	38	38, 40
19	20	20, 22	39	56	

- Red: touchable numbers of the form $s(p^2) \leq X$ for p prime;
- Green: touchable numbers of the form $s(2^j n) \leq X$ for $n \neq \square$;
- Blue: touchable numbers of the form $s(n^2) \leq X$ for n composite and $\leq X^{2/3}$;
- The only untouchable numbers up to 40 are 2 and 5.

Pomerance-Yang Algorithm on the Larger Scale

- Let K be the number of files (K divides X). Each file contains touchable numbers from $kX/K+2$ to $(k+1)X$ for $k=1,2,\ldots,K$.
- Compute $s(n)$ using the Pomerance-Yang Algorithm (in parallel). For each $s(n)$ determine k such that

$$kX/K+2 \leq s(n) \leq (k+1)X/K$$

  and write $s(n)$ into a k-th buffer.
- When the k-th buffer gets filled, write its contents into the k-th file.
- Run the computation of $s(n^2)$ for composite $n \leq X^{2/3}$ separately.

Counts of Untouchable Numbers to 2^{40}

- $U(X)$ denotes the total count of untouchable numbers $\leq X$.

X	$U(X)$	$U(X)/X$	X	$U(X)$	$U(X)/X$
10^{11}	16988116409	0.1699	$7 \cdot 10^{11}$	119670797251	0.1710
$2 \cdot 10^{11}$	34059307043	0.1703	$8 \cdot 10^{11}$	136818383894	0.1710
$3 \cdot 10^{11}$	51156680233	0.1705	$9 \cdot 10^{11}$	153971157176	0.1711
$4 \cdot 10^{11}$	68270208722	0.1707	10^{12}	171128671374	0.1711
$5 \cdot 10^{11}$	85395279511	0.1708	2^{40}	188206399403	0.1712
$6 \cdot 10^{11}$	102529360015	0.1709			

- For $X = 2^{40}$, $\frac{1}{\lfloor X/2 \rfloor - U(X) + 1} \sum_{\substack{\text{even } n \leq X \\ n \text{ is touchable}}} \log \frac{s(n)}{n} \approx -0.08852.$

- For $X = 2^{40}$, $\frac{1}{U(X) - 1} \sum_{\substack{\text{even } n \leq X \\ n \text{ is untouchable}}} \log \frac{s(n)}{n} \approx 0.07290.$

Part V.
On the Tabulation of k-untouchable Numbers

Background

- Let k be a positive integer. A number n is called *k-untouchable* if there is no m such that $n = s_k(m)$.
- Note that if a number is k-untouchable, it is $(k+1)$-untouchable, $(k+2)$-untouchable and so on.
- All k-untouchable numbers occur in the aliquot sequences which start with an untouchable number.

Example

- First aliquot sequences which start with an untouchable number (excluding 2 and 5):

52	46	26	16	15
88	92	76	64	63
96	156	236	184	176
120	240	504	1056	1968

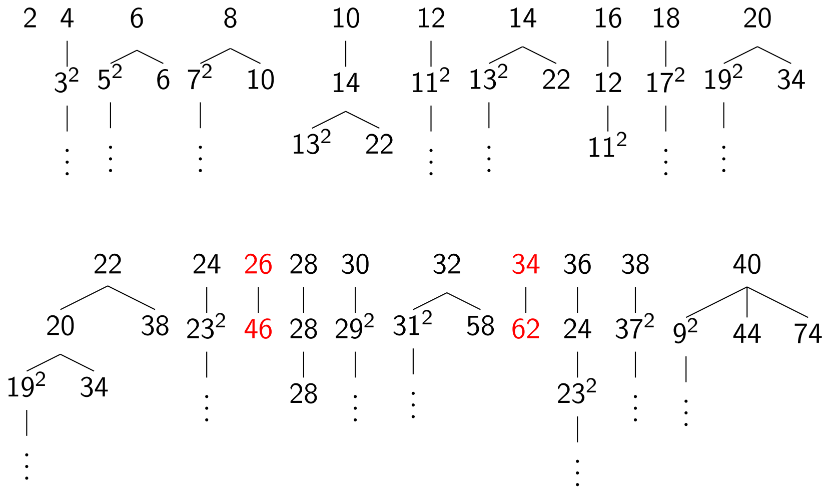
- For example, 46 is a candidate for a 2-untouchable number. However, $46 = s(86) = s_2(166)$, so 46 is not 2-untouchable.
- In fact, the first 2-untouchable number which is not untouchable is 208.
- We propose a simple recursive algorithm to tabulate k-untouchable numbers for all $1 \leq k \leq K$ and even $n \leq X$.
- Our algorithm assumes that the variant of a Goldbach's conjecrure discussed above is true.

Example for $k \leq 2$ and $X \leq 40$

When using the Pomerance-Yang Algorithm, along with the touchable numbers we will also store their preimages:

Touchable number	Preimages
2	
4	3^2
6	5^2, 6
8	7^2, 10
10	14
12	11^2
14	13^2, 22
16	12
18	17^2
20	19^2, 34
22	20, 38
24	23^2
26	46
28	28
30	29^2
32	31^2, 58
34	62
36	24
38	37^2
40	9^2, 44, 50

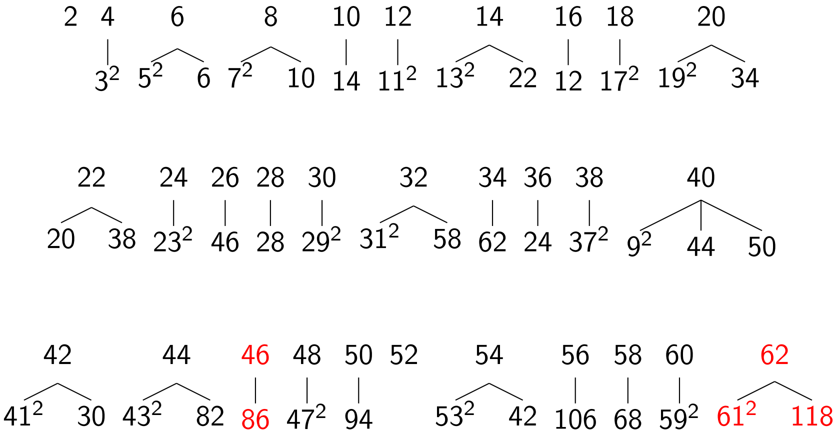
Example for $k \leq 2$ and $X \leq 40$



To determine whether 26 and 34 are 2-touchable, we need to compute the preimages of 46 and 62 under s.

Example for $k \leq 2$ and $X \leq 40$

We use the Pomeance-Yang Algorithm again to expand our table of touchable numbers to 62:



Thus all the numbers up to 40, except for 2 and 5, are 2-touchable.

To Do List

- Up to some bound X, tabulate all the even k-untouchable numbers and compute

$$\frac{1}{\lfloor X/2 \rfloor - U_k(X) + 1} \sum_{\substack{\text{even } n \leq X \\ n \text{ is } k\text{-touchable}}} \log \frac{s(n)}{n},$$

  where $U_k(X)$ denotes the total number of k-untouchable numbers up to X. Will this influence Guy's heuristics?
- Perhaps, weighted sums makes more sense? For example,

$$s(192) = s(304) = s(344) = s(412) = 316,$$

  so the number 316 should be considered with the weight 4, while untouchable numbers should be assigned weight zero.
- Come up with the heuristic argument for the density of the k-untouchable numbers.

Happy 100th Birthday, Professor Guy!