Math 1410–Solutions for Assignment 9

Submitted Friday, December 2, 2005

1. (a) Verify that the three vectors $\underline{u} = \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$, $\underline{v} = \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{-1}{\sqrt{3}}\right)$, $\underline{w} = \left(\frac{-1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$ form an orthonormal basis for $\mathbb{R}^3$.

<u>**Solution**</u>**:**

We need to show that each pair of vectors is orthogonal and that each vector has a length/magnitude/norm of 1. There are two ways of doing this.

Method 1: Verify the orthogonality of each pair of vectors separately and calculate the length of each vector separately.

$$\underline{u} \circ \underline{v} = \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{3}}\right) + 0 + \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{-1}{\sqrt{3}}\right) = \tfrac{1}{\sqrt{6}} - \tfrac{1}{\sqrt{6}} = 0.$$

$$\underline{u} \circ \underline{w} = \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{-1}{\sqrt{6}}\right) + 0 + \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{6}}\right) = \tfrac{-1}{\sqrt{12}} + \tfrac{1}{\sqrt{12}} = 0.$$

$$\underline{v} \circ \underline{w} = \tfrac{1}{\sqrt{3}} \cdot \tfrac{-1}{\sqrt{6}} + \tfrac{1}{\sqrt{3}} \cdot \tfrac{2}{\sqrt{6}} + \tfrac{-1}{\sqrt{3}} \cdot \tfrac{1}{\sqrt{6}} = \tfrac{-1}{\sqrt{18}} + \tfrac{2}{\sqrt{18}} - \tfrac{1}{\sqrt{18}} = 0.$$

$$||\underline{u}|| = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + (0)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}} = 1.$$

$$||\underline{v}|| = \sqrt{\left(\tfrac{1}{\sqrt{3}}\right)^2 + \left(\tfrac{1}{\sqrt{3}}\right)^2 + \left(\tfrac{-1}{\sqrt{3}}\right)^2} = \sqrt{\tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3}} = 1.$$

$$||\underline{w}|| = \sqrt{\left(\tfrac{-1}{\sqrt{6}}\right)^2 + \left(\tfrac{2}{\sqrt{6}}\right)^2 + \left(\tfrac{1}{\sqrt{6}}\right)^2} = \sqrt{\tfrac{1}{6} + \tfrac{4}{6} + \tfrac{1}{6}} = 1.$$

Therefore, $\{\underline{u},\ \underline{v},\ \underline{w}\}$ is an orthonormal basis for $\mathbb{R}^3$.

Method 2: Form a 3×3 matrix M whose rows are the three vectors and verify that $MM^t = I$.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{12}} + \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} & \frac{1}{3} + \frac{1}{3} + \frac{1}{3} & \frac{-1}{\sqrt{18}} + \frac{2}{\sqrt{18}} - \frac{1}{\sqrt{18}} \\ \frac{-1}{\sqrt{12}} + \frac{1}{\sqrt{12}} & \frac{-1}{\sqrt{18}} + \frac{2}{\sqrt{18}} - \frac{1}{\sqrt{18}} & \frac{1}{6} + \frac{4}{6} + \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, $\{\underline{u},\ \underline{v},\ \underline{w}\}$ is an orthonormal basis for $\mathbb{R}^3$.

(b) Express the vectors $(1, -3,\ 4)$ and $(2,\ 1,\ 2)$ as linear combinations of the above basis.

Solution:

Let $\underline{a} = (1, -3,\ 4)$ and $\underline{b} = (2,\ 1,\ 2)$. Then,

$$\begin{aligned} \underline{a} &= (\underline{a} \circ \underline{u})\ \underline{u} + (\underline{a} \circ \underline{v})\ \underline{v} + (\underline{a} \circ \underline{w})\ \underline{w} \\ &= \left(\frac{1}{\sqrt{2}} + 0 + \frac{4}{\sqrt{2}}\right) \underline{u} + \left(\frac{1}{\sqrt{3}} - \frac{3}{\sqrt{3}} - \frac{4}{\sqrt{3}}\right) \underline{v} + \left(\frac{-1}{\sqrt{6}} - \frac{6}{\sqrt{6}} + \frac{4}{\sqrt{6}}\right) \underline{w} \\ &= \frac{5}{\sqrt{2}}\,\underline{u} + \frac{-6}{\sqrt{3}}\,\underline{v} + \frac{-3}{\sqrt{6}}\,\underline{w}. \end{aligned}$$

Similarly,

$$\begin{aligned}
\underline{b} &= (\underline{b} \circ \underline{u})\, \underline{u} + (\underline{b} \circ \underline{v})\, \underline{v} + (\underline{b} \circ \underline{w})\, \underline{w} \\
&= \left(\frac{2}{\sqrt{2}} + 0 + \frac{2}{\sqrt{2}}\right) \underline{u} + \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}}\right) \underline{v} + \left(\frac{-2}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}}\right) \underline{w} \\
&= \frac{4}{\sqrt{2}}\, \underline{u} + \frac{1}{\sqrt{3}}\, \underline{v} + \frac{2}{\sqrt{6}}\, \underline{w}.
\end{aligned}$$

2. (a) Use the Gram-Schmidt process to orthonormalize the vectors

$$(1,\ 1,\ 1,\ 1),\ (1,\ 1,\ 1, -1),\ (1,\ 2,\ 2,\ 0).$$

Solution:

Let $\underline{v}_1 = (1,\ 1,\ 1,\ 1)$, $\underline{v}_2 = (1,\ 1,\ 1, -1)$ and $\underline{v}_3 = (1,\ 2,\ 2,\ 0)$, and let $S = \{\underline{v}_1,\ \underline{v}_1,\ \underline{v}_1\}$. Then, we define $\underline{u}_1$, $\underline{u}_2$, and $\underline{u}_3$ by

$$\begin{aligned}
\underline{u}_1 &= \underline{v}_1 = (1,\ 1,\ 1,\ 1), \\
\underline{u}_2 &= \underline{v}_2 - \text{proj}_{\underline{u}_1} \underline{v}_2 = \underline{v}_2 - \frac{\underline{v}_2 \circ \underline{u}_1}{\underline{u}_1 \circ \underline{u}_1}\, \underline{u}_1 \\
&= (1,\ 1,\ 1, -1) - \frac{1+1+1-1}{1+1+1+1}\, (1,\ 1,\ 1,\ 1) \\
&= (1,\ 1,\ 1, -1) - \left(\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\right) \\
&= \left(\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}, -\tfrac{3}{2}\right), \quad \text{and} \\
\underline{u}_3 &= \underline{v}_3 - \text{proj}_{\underline{u}_1} \underline{v}_3 - \text{proj}_{\underline{u}_2} \underline{v}_3 \\
&= \underline{v}_3 - \frac{\underline{v}_3 \circ \underline{u}_1}{\underline{u}_1 \circ \underline{u}_1}\, \underline{u}_1 - \frac{\underline{v}_3 \circ \underline{u}_2}{\underline{u}_2 \circ \underline{u}_2}\, \underline{u}_2 \\
&= (1,\ 2,\ 2,\ 0) - \tfrac{5}{4}\, (1,\ 1,\ 1,\ 1) - \tfrac{5}{6} \left(\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}, -\tfrac{3}{2}\right) \\
&= (1,\ 2,\ 2,\ 0) - \left(\tfrac{5}{4},\ \tfrac{5}{4},\ \tfrac{5}{4},\ \tfrac{5}{4}\right) - \left(\tfrac{5}{12},\ \tfrac{5}{12},\ \tfrac{5}{12}, -\tfrac{5}{4}\right) \\
&= \left(-\tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{1}{3},\ 0\right).
\end{aligned}$$

We have obtained an orthogonal basis $\{\underline{u}_1,\ \underline{u}_2,\ \underline{u}_3\}$. We now find an orthonormal basis $\{\underline{w}_1,\ \underline{w}_2,\ \underline{w}_3\}$ by normalizing $\underline{u}_1$, $\underline{u}_2$, and $\underline{u}_3$.

This computation may be simplified by first finding (non-zero) scalar multiples of $\underline{u}_1$, $\underline{u}_2$, and $\underline{u}_3$ that have integer components and then normalizing these new vectors. In other words, we can normalize $\underline{q}_1 = \underline{u}_1 = (1,\ 1,\ 1,\ 1)$, $\underline{q}_2 = 2\underline{u}_2 = (1,\ 1,\ 1, -3)$, and $\underline{q}_3 = 3\underline{u}_3 = (-2,\ 1,\ 1,\ 0)$ to obtain

$$\underline{w}_1 = \frac{1}{||\underline{q}_1||}\,\underline{q}_1 = \frac{1}{\sqrt{4}}\,(1,\ 1,\ 1,\ 1) = \left(\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\right),$$

$$\underline{w}_2 = \frac{1}{||\underline{q}_2||}\,\underline{q}_2 = \frac{1}{\sqrt{12}}\,(1,\ 1,\ 1, -3) = \left(\frac{1}{\sqrt{12}},\ \frac{1}{\sqrt{12}},\ \frac{1}{\sqrt{12}},\ \frac{-3}{\sqrt{12}}\right),$$

$$\text{and } \underline{w}_3 = \frac{1}{||\underline{q}_3||}\,\underline{q}_3 = \frac{1}{\sqrt{6}}\,(-2,\ 1,\ 1,\ 0) = \left(\frac{-2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ 0\right).$$

Then, $\{\underline{w}_1,\ \underline{w}_2,\ \underline{w}_3\}$ is an orthonormal basis of the span of S.

(b) Use part (a) to see if the vector $(1,\ 2,\ 2, -2)$ is in $\text{span}\{(1,\ 1,\ 1,\ 1),\ (1,\ 1,\ 1, -1),\ (1,\ 2,\ 2,\ 0)\}$.

Solution:

Let $\underline{a} = (1,\ 2,\ 2, -2)$ and $\underline{b} = (1,\ 2,\ 4, -2)$.

There are two ways of using part (a) to determine which of $\underline{a}$ and $\underline{b}$ are in the span of S. For part (b), one method will be used, and for part (c), the other method will be used.

We begin by calculating the projection of $\underline{a}$ onto the span of S using its orthonormal basis $\{\underline{w}_1,\ \underline{w}_2,\ \underline{w}_3\}$:

$$
\begin{aligned}
& \operatorname{proj}_{\underline{w}_1}\underline{a} + \operatorname{proj}_{\underline{w}_2}\underline{a} + \operatorname{proj}_{\underline{w}_3}\underline{a} \\
= \ & (\underline{a} \circ \underline{w}_1)\ \underline{w}_1 + (\underline{a} \circ \underline{w}_2)\ \underline{w}_2 + (\underline{a} \circ \underline{w}_3)\ \underline{w}_3 \\
= \ & \tfrac{3}{2}\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) + \tfrac{11}{\sqrt{12}}\left(\tfrac{1}{\sqrt{12}}, \tfrac{1}{\sqrt{12}}, \tfrac{1}{\sqrt{12}}, \tfrac{-3}{\sqrt{12}}\right) + \tfrac{2}{\sqrt{6}}\left(\tfrac{-2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, 0\right) \\
= \ & \left(\tfrac{3}{4},\ \tfrac{3}{4},\ \tfrac{3}{4},\ \tfrac{3}{4}\right) + \left(\tfrac{11}{12},\ \tfrac{11}{12},\ \tfrac{11}{12}, -\tfrac{11}{4}\right) + \left(-\tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{1}{3},\ 0\right) \\
= \ & (1,\ 2,\ 2, -2)\,.
\end{aligned}
$$

Since $\underline{a}$ is equal to its own projection onto the span of S, $\underline{a}$ must be *in* the span of S.

(c) Repeat part (b) for the vector $(1,\ 2,\ 4, -2)$.

Solution:

This time we calculate the projection of of $\underline{b}$ onto the span of S using its *orthogonal* basis $\left\{\underline{q}_1,\ \underline{q}_2,\ \underline{q}_3\right\}$:

$$
\begin{aligned}
& \operatorname{proj}_{\underline{q}_1}\underline{b} + \operatorname{proj}_{\underline{q}_2}\underline{b} + \operatorname{proj}_{\underline{q}_3}\underline{b} \\
= \ & \frac{\underline{b} \circ \underline{q}_1}{\underline{q}_1 \circ \underline{q}_1}\,\underline{q}_1 + \frac{\underline{b} \circ \underline{q}_2}{\underline{q}_2 \circ \underline{q}_2}\,\underline{q}_2 + \frac{\underline{b} \circ \underline{q}_3}{\underline{q}_3 \circ \underline{q}_3}\,\underline{q}_3 \\
= \ & \tfrac{5}{4}\,(1,\ 1,\ 1,\ 1) + \tfrac{13}{12}\,(1,\ 1,\ 1, -3) + \tfrac{4}{\sqrt{6}}\,(-2,\ 1,\ 1,\ 0) \\
= \ & \left(\tfrac{5}{4},\ \tfrac{5}{4},\ \tfrac{5}{4},\ \tfrac{5}{4}\right) + \left(\tfrac{13}{12},\ \tfrac{13}{12},\ \tfrac{13}{12}, -\tfrac{13}{4}\right) + \left(-\tfrac{4}{3},\ \tfrac{2}{3},\ \tfrac{2}{3},\ 0\right) \\
= \ & (1,\ 3,\ 3, -2)\,.
\end{aligned}
$$

Since $\underline{b}$ is not equal to its projection onto the span of S, $\underline{b}$ is not in the span of S.

3. (a) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}.$$

<u>Solution</u>:

The eigenvalues of A are the values of the scalar λ for which the equation $A\underline{x} = \lambda\underline{x}$ has a non-zero solution for the column vector $\underline{x}$.

The equation above may be rewritten as $A\underline{x} - \lambda\underline{x} = 0$, which in turn becomes $(A - \lambda I)\underline{x} = 0$. For a fixed value of λ, this equation represents a homogeneous linear system. This system will have a non-zero solution exactly when its coefficient matrix is not invertible. In other words, there is a non-zero solution for $\underline{x}$ exactly when the determinant of $A - \lambda I$ is 0.

Consequently, to find the eigenvalues of A, we calculate the determinant of $A - \lambda I$, set it equal to 0, then solve for λ.

$$\begin{aligned}
&|A - \lambda I| \\
&= \left| \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| \\
&= \left| \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| \\
&= \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 3 & 1-\lambda \end{vmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= (3-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} \\
&= (3-\lambda)\left[(1-\lambda)^2 - 3^2\right] \\
&= (3-\lambda)\left[1-\lambda-\lambda-\lambda^2-9\right] \\
&= (3-\lambda)\left[\lambda^2-2\lambda-8\right] \\
&= (3-\lambda)\left[(\lambda-4)(\lambda+2)\right].
\end{aligned}
$$

The above expression, which is a polynomial in λ, is equal to 0 when λ is 3, 4, or -2. Thus, the eigenvalues of A are 3, 4, and -2.

(b) Find a basis for each of the eigenspaces of the matrix A.

Solution:

An eigenspace of a matrix B is the solution set of the linear system $(B-\lambda I)\underline{x} = 0$, where λ is an eigenvalue of B. Since the matrix A above has three eigenvalues, it will have three eigenspaces.

To find the basis of each eigenspace of A, we solve each of the three linear systems obtained by replacing λ in the equation $(A-\lambda I)\underline{x} = 0$ with an eigenvalue of A.

$\underline{\lambda = 3}$: The augmented matrix for the system $(A-3I)\underline{x} = 0$ is

$$
\left[\begin{array}{ccc|c} 3-(3) & 0 & 0 & 0 \\ 0 & 1-(3) & 3 & 0 \\ 0 & 3 & 1-(3) & 0 \end{array}\right]
$$

$$
\begin{aligned}
&= \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 3 & -2 & 0 \end{array}\right] \\
&\underset{R1 \longleftrightarrow R3}{\sim} \left[\begin{array}{ccc|c} 0 & 3 & -2 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\
&\underset{R2+R1}{\sim} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\
&\underset{2R1+R2}{\sim} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\
&\underset{\frac{1}{5}R2}{\sim} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\
&\underset{-R2+R1}{\sim} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].
\end{aligned}
$$

The general solution to this system is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,
so the basis of this eigenspace is $\{(1,\ 0,\ 0)\}$.

$\underline{\lambda = 4}$: The augmented matrix for the system $(A - 4I)\underline{x} = 0$ is

$$
\left[\begin{array}{ccc|c} 3-(4) & 0 & 0 & 0 \\ 0 & 1-(4) & 3 & 0 \\ 0 & 3 & 1-(4) & 0 \end{array}\right]
$$

$$
\begin{aligned}
&= \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \\
&\underset{R2+R3}{\sim} \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\
&\underset{\substack{-R1 \\ -\frac{1}{3}R1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].
\end{aligned}
$$

The general solution to this system is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ z \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$,
so the basis of this eigenspace is $\{(1,\ 0,\ 0)\}$.

$\underline{\lambda = -2}$: The augmented matrix for the system $(A-(-2)I)\underline{x} = 0$ is

$$
\begin{aligned}
&\left[\begin{array}{ccc|c} 3-(-2) & 0 & 0 & 0 \\ 0 & 1-(-2) & 3 & 0 \\ 0 & 3 & 1-(-2) & 0 \end{array}\right] \\
&= \left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \\
&\underset{-R2+R3}{\sim} \left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\
&\underset{\substack{\frac{1}{5}R1 \\ \frac{1}{3}R2}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].
\end{aligned}
$$

The general solution to this system is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$,
so the basis of this eigenspace is $\{(0, -1,\ 1)\}$.

(c) Orthonormalize the vectors found in (b) by applying the Gram-Schmidt process, if necessary.

Solution:

The vectors $(1,\ 0,\ 0)$, $(0, -1,\ 1)$, and $(0,\ 1,\ 1)$ are already orthogonal, so we do not need to use the Gram-Schmidt process. We do need to normalize these vectors, however:

$$\frac{1}{||(1,\ 0,\ 0)||}\,(1,\ 0,\ 0) = \frac{1}{1}\,(1,\ 0,\ 0) = (1,\ 0,\ 0),$$

$$\frac{1}{||(0, -1,\ 1)||}\,(0, -1,\ 1) = \frac{1}{\sqrt{2}}\,(0, -1,\ 1) = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

and $$\frac{1}{||(0,\ 1,\ 1)||}\,(0,\ 1,\ 1) = \frac{1}{\sqrt{2}}\,(0,\ 1,\ 1) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

(d) Use the vectors found in (c) to form an orthonormal matrix P diagonalizing A.

Solution:

P is a 3×3 matrix whose columns are the vectors found in (c) i.e.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

We now verify that P diagonalizes A:

$$P^tAP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^t \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \\ 0 & \frac{-2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D.$$

D is a diagonal matrix, so P diagonalizes A.

(e) Find the entry in the first row and first column of A^7.

Solution:

Since $P^tP = I$ and P is square, $P^{-1} = P^t$. Then, $P^tAP = D$

$$\Longrightarrow \quad P\left(P^tAP\right)P^t = P\left(D\right)P^t$$

$$\Longrightarrow \quad A = PDP^t$$

$$\Longrightarrow \quad A^2 = \left(PDP^t\right)\left(PDP^t\right) = PD^2P^t$$

$$\Longrightarrow \quad A^3 = \left(PD^2P^t\right)\left(PDP^t\right) = PD^3P^t, \text{ etc.}$$

Continuing, we obtain $A^n = PD^nP^t$. Thus, $A^7 = PD^7P^t$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}^7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3^7 & 0 & 0 \\ 0 & -2^7 & 0 \\ 0 & 0 & 4^7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3^7 & 0 & 0 \\ 0 & \frac{2^7}{\sqrt{2}} & \frac{-2^7}{\sqrt{2}} \\ 0 & \frac{4^7}{\sqrt{2}} & \frac{4^7}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 3^7 & 0 & 0 \\ 0 & \frac{-2^7+4^7}{2} & \frac{2^7+4^7}{2} \\ 0 & \frac{2^7+4^7}{2} & \frac{-2^7+4^7}{2} \end{bmatrix}.$$

So, the entry in the first row and first column of A^7 is 3^7, or 2187.

4. Repeat Problem 3 for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

<u>Solution</u>:

As in Problem 3, we calculate the determinant of $A - \lambda I$, set it equal to 0, then solve for λ.

$$\begin{aligned}
&|A - \lambda I| \\
&= \left| \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| \\
&= \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} \\
&= (-1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} \\
&= (-1-\lambda)\left[(1-\lambda)^2 - 2^2\right] \\
&= (-1-\lambda)\left[(1-\lambda-2)(1-\lambda+2)\right] \\
&= (-1-\lambda)\left[(-1-\lambda)(3-\lambda)\right] \\
&= (1+\lambda)^2(3-\lambda).
\end{aligned}$$

The above polynomial is equal to 0 when λ is -1 or 3. Thus, the eigenvalues of A are -1 and 3.

(b) Find a basis for each of the eigenspaces of the matrix A.

Solution:

As in Problem 3, we find the basis of each eigenspace of A by solving both of the linear systems obtained by replacing λ in the equation $(A-\lambda I)\underline{x} = 0$ with an eigenvalue of A.

$\underline{\lambda = -1}$: The augmented matrix for the system $(A-(-1)I)\underline{x} = 0$ is

$$\left[\begin{array}{ccc|c} 1-(-1) & 2 & 0 & 0 \\ 2 & 1-(-1) & 0 & 0 \\ 0 & 0 & -1-(-1) & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{\frac{1}{2}\text{R1}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{-2\text{R1}+\text{R2}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution to this system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

so the basis of this eigenspace is $\{(-1,\ 1,\ 0),\ (0,\ 0,\ 1)\}$.

$\underline{\lambda = 3}$: The augmented matrix for the system $(A-3I)\underline{x} = 0$ is

$$\left[\begin{array}{ccc|c} 1-(3) & 2 & 0 & 0 \\ 2 & 1-(3) & 0 & 0 \\ 0 & 0 & -1-(3) & 0 \end{array}\right]$$

$$= \left[\begin{array}{rrr|r} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

$$\underset{\substack{-\frac{1}{2}\mathsf{R1} \\ -\frac{1}{4}\mathsf{R3}}}{\sim} \left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{-2\mathsf{R1}+\mathsf{R2}}{\sim} \left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{\mathsf{R2} \longleftrightarrow \mathsf{R3}}{\sim} \left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution to this system is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$,

so the basis of this eigenspace is $\{(1,\ 1,\ 0)\}$.

(c) Orthonormalize the vectors found in (b) by applying the Gram-Schmidt process, if necessary.

Solution:

Like the vectors in Problem 3, the vectors $(-1,\ 1,\ 0)$, $(0,\ 0,\ 1)$, and $(1,\ 1,\ 0)$ are already orthogonal, so we just need to normalize them:

$$\frac{1}{||(-1,\ 1,\ 0)||}\,(-1,\ 1,\ 0) \;=\; \frac{1}{\sqrt{2}}\,(-1,\ 1,\ 0) \;=\; \left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),$$

$$\frac{1}{||(0,\ 0,\ 1)||}\,(0,\ 0,\ 1) \;=\; \frac{1}{1}\,(0,\ 0,\ 1) \;=\; (0,\ 0,\ 1),\ \text{ and}$$

$$\frac{1}{||(1,\ 1,\ 0)||}\,(1,\ 1,\ 0) \;=\; \frac{1}{\sqrt{2}}\,(1,\ 1,\ 0) \;=\; \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right).$$

(d) Use the vectors found in (c) to form an orthonormal matrix P diagonalizing A.

Solution:

As in Problem 3, the columns of P are the vectors found in (c) i.e.

$$P \;=\; \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}.$$

We now verify that P diagonalizes A:

$$P^tAP = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}^t \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{3}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{3}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D.$$

D is a diagonal matrix, so P diagonalizes A.

(e) Find the entry in the first row and first column of A^7.

Solution:

As shown in Problem 3, $A^n = PD^nP^t$. Thus, $A^7 = PD^7P^t$

$$= \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}^7 \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}^t$$

$$= \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3^7 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{3^7}{\sqrt{2}} & \frac{3^7}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1+3^7}{2} & \frac{1+3^7}{2} & 0 \\ \frac{1+3^7}{2} & \frac{-1+3^7}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

So, the entry in the first row and first column of A^7 is

$$\frac{-1+3^7}{2} = \frac{-1+2187}{2} = \frac{2186}{2} = 1093.$$