Solutions for Assignment 1

Submitted Friday, September 23

Solve the following two systems by using Gaussian elimination:

1.

$$\begin{array}{ccccccc} x & + & 3y & + & z & = & 4 \\ & & 9y & + & z & = & 2 \\ & & & & z & = & 2 \end{array}$$

<u>**Solution**</u>**:**

The system is already in triangular form, so we are ready to do back substitution. This can either be done with elementary operations or without elementary operations.

Method 1: To avoid using elementary operations, we substitute the value of z from the third equation (i.e. $z = 2$) into the second equation to solve for y:

$$9y + (2) = 2 \Longrightarrow 9y = 2 - 2 = 0 \Longrightarrow y = \frac{0}{9} = 0.$$

Next, we substitute the known values of y and z (i.e. $y = 0$ and $z = 2$) into the first equation to solve for x:

$$x + 3(0) + (2) = 4 \Longrightarrow x = 4 - 2 = 2.$$

So, the solution is $x = 2$, $y = 0$, and $z = 2$.

Method 2: Using elementary operations, we use the third equation to eliminate z from the first and second equations, then we turn the coefficient of y in the second equation into a 1, and finally we use the second equation to eliminate y from the first equation, as follows:

$$\begin{array}{c} \sim \\ -\mathsf{R3} + \mathsf{R1} \\ -\mathsf{R3} + \mathsf{R2} \end{array} \quad \left\{ \begin{array}{ccccccc} x & + & 3y & & & = & 2 \\ & & 9y & & & = & 0 \\ & & & & z & = & 2 \end{array} \right.$$

$$\begin{array}{c} \sim \\ \frac{1}{9}\mathsf{R2} \end{array} \quad \left\{ \begin{array}{ccccccc} x & + & 3y & & & = & 2 \\ & & y & & & = & 0 \\ & & & & z & = & 2 \end{array} \right.$$

$$\begin{array}{c} \sim \\ -3\mathsf{R2} + \mathsf{R1} \end{array} \quad \left\{ \begin{array}{ccccc} x & & & = & 2 \\ & y & & = & 0 \\ & & z & = & 2 \end{array} \right. .$$

2.

$$\begin{array}{rcrcrcr} x & + & 5y & + & 4z & = & 10 \\ 2x & - & y & + & z & = & 2 \\ x & + & y & + & 3z & = & 5 \end{array}$$

<u>Solution</u>:

$$\begin{array}{c} \sim \\ -2\text{R1}+\text{R2} \\ -\text{R1}+\text{R3} \end{array} \quad \left\{ \begin{array}{rcrcrcr} x & + & 5y & & 4z & = & 10 \\ & - & 11y & - & 7z & = & -18 \\ & - & 4y & - & z & = & -5 \end{array} \right.$$

$$\begin{array}{c} \sim \\ -3\text{R3}+\text{R2} \end{array} \quad \left\{ \begin{array}{rcrcrcr} x & + & 5y & & 4z & = & 10 \\ & & y & - & 4z & = & -3 \\ & - & 4y & - & z & = & -5 \end{array} \right.$$

$$\begin{array}{c} \sim \\ 4\text{R2}+\text{R3} \end{array} \quad \left\{ \begin{array}{rcrcrcr} x & + & 5y & & 4z & = & 10 \\ & & y & - & 4z & = & -3 \\ & & & - & 17z & = & -17 \end{array} \right.$$

$$\begin{array}{c} \sim \\ -\frac{1}{17}\text{R3} \end{array} \quad \left\{ \begin{array}{rcrcrcr} x & + & 5y & & 4z & = & 10 \\ & & y & - & 4z & = & -3 \\ & & & & z & = & 1 \end{array} \right.$$

$$\begin{array}{c} \sim \\ -4\text{R3}+\text{R1} \\ 4\text{R3}+\text{R2} \end{array} \quad \left\{ \begin{array}{rcrcrcr} x & + & 5y & & & = & 6 \\ & & y & & & = & 1 \\ & & & & z & = & 1 \end{array} \right.$$

$$\begin{array}{c} \sim \\ -5\text{R2}+\text{R1} \end{array} \quad \left\{ \begin{array}{rcrcrcr} x & & & & & = & 1 \\ & & y & & & = & 1 \\ & & & & z & = & 1 \end{array} \right. .$$

3. Determine if the system below is consistent.

$$\begin{array}{rcrcrcrcr} x & + & 6y & + & z & & & = & 8 \\ & & y & - & 3z & + & t & = & -2 \\ x & - & y & + & z & + & 3t & = & 1 \\ x & + & y & - & 5z & - & 4t & = & 0 \end{array}$$

Solution:

$$\begin{array}{c} \sim \\ -\mathsf{R1}+\mathsf{R3} \\ -\mathsf{R1}+\mathsf{R4} \end{array} \left\{ \begin{array}{rcrcrcrcr} x & + & 6y & + & z & & & = & 8 \\ & & y & - & 3z & + & t & = & -2 \\ & - & 7y & & & + & 3t & = & -7 \\ & - & 5y & - & 6z & - & 4t & = & -8 \end{array} \right.$$

$$\begin{array}{c} \sim \\ 7\mathsf{R2}+\mathsf{R3} \\ 5\mathsf{R2}+\mathsf{R4} \end{array} \left\{ \begin{array}{rcrcrcrcr} x & + & 6y & + & z & & & = & 8 \\ & & y & - & 3z & + & t & = & -2 \\ & & & - & 21z & + & 10t & = & -21 \\ & & & - & 21z & + & t & = & -18 \end{array} \right.$$

$$\begin{array}{c} \sim \\ -\mathsf{R3}+\mathsf{R4} \end{array} \left\{ \begin{array}{rcrcrcrcr} x & + & 6y & + & z & & & = & 8 \\ & & y & - & 3z & + & t & = & -2 \\ & & & - & 21z & + & 10t & = & -21 \\ & & & & & - & 9t & = & 3 \end{array} \right.$$

$$\begin{array}{c} \sim \\ -\frac{1}{21}\mathsf{R3} \\ \\ -\frac{1}{9}\mathsf{R4} \end{array} \left\{ \begin{array}{rcrcrcrcr} x & + & 6y & + & z & & & = & 8 \\ & & y & - & 3z & + & t & = & -2 \\ & & & & z & - & \frac{10}{21}t & = & 1 \\ & & & & & & t & = & -\frac{1}{3} \end{array} \right. .$$

The system is now in echelon form. Since there is no contradictory equation (such as $0 = 1$), the system is consistent.

In the next two questions, the augmented matrix of a system of linear equations in unknowns x,y,z,t is shown. In each case, find the solution set of the system.

4.

$$\left[\begin{array}{rrrr|r} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 \end{array}\right]$$

<u>Solution</u>:

$$\underset{-2\text{R3}+\text{R2}}{\sim} \left[\begin{array}{rrrr|r} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 4 \\ 0 & 0 & 1 & 1 & -2 \end{array}\right]$$

$$\underset{-3\text{R2}+\text{R1}}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 10 & -12 \\ 0 & 1 & 0 & -3 & 4 \\ 0 & 0 & 1 & 1 & -2 \end{array}\right].$$

This matrix is in reduced echelon form. We note that there are leading ones in the columns representing the coefficients of x, y, and z, but there is no leading one in the column representing the coefficients of t. We conclude that x, y, and z are dependent variables while t is arbitrary. Consequently, we set t equal to itself and solve the remaining linear equations for x, y, and z to obtain the general solution:

$$\begin{cases} x &= -12-10t \\ y &= 4+3t \\ z &= -2-t \\ t &= t \end{cases}.$$

5.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & -5 \\ 0 & 1 & -2 & 1 & -7 \\ 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

<u>Solution</u>:

$$\underset{\substack{-2R4+R1 \\ -R4+R2 \\ -4R4+R3}}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -13 \\ 0 & 1 & -2 & 0 & -11 \\ 0 & 0 & 1 & 0 & -18 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$\underset{2R3+R2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -13 \\ 0 & 1 & 0 & 0 & -47 \\ 0 & 0 & 1 & 0 & -18 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$\underset{-R2+R1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 34 \\ 0 & 1 & 0 & 0 & -47 \\ 0 & 0 & 1 & 0 & -18 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

So, the solution is

$$\begin{cases} x = 34 \\ y = -47 \\ z = -18 \\ t = 4 \end{cases}.$$

6. Determine the value of h that makes the following matrix the augmented matrix of a consistent linear system.

$$\left[\begin{array}{cc|c} 1 & 4 & -2 \\ 5 & h & -5 \end{array}\right]$$

Solution:

$$\underset{-5\text{R1}+\text{R2}}{\sim} \left[\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & (h-20) & 5 \end{array}\right].$$

If $h = 20$, then the matrix becomes

$$\left[\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & 0 & 5 \end{array}\right],$$

which is the augmented matrix of an inconsistent system.

If $h \neq 20$, the system is consistent. We can stop here, but it is better to continue as follows:

If $h \neq 20$, then we can do back substitution:

$$(h-20)y = 5 \Longrightarrow y = \frac{5}{h-20}.$$

Then, $x + 4y = -2$

$$\Longrightarrow x = -2 - 4y = -2 - 4\left(\frac{5}{h-20}\right) = \frac{-2(h-20)}{h-20} - \frac{20}{h-20} = \frac{20-2h}{h-20}.$$

So, the unique solution in this case is

$$\begin{cases} x = \frac{20-2h}{h-20} \\ y = \frac{5}{h-20} \end{cases}.$$

Ergo, the system is consistent when $h \neq 20$.

7. Show that the three planes $x+y+z=3$, $2x-y+z=2$, and $2x-3y+z=0$ intersect at a point and find that point of intersection.

Solution:

We first form the linear system containing the equations of the three planes:

$$\begin{cases} x + y + z = 3 \\ 2x - y + z = 2 \\ 2x - 3y + z = 0 \end{cases}.$$

We then solve this system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & 1 & 2 \\ 2 & -3 & 1 & 0 \end{array}\right]$$

$$\underset{\substack{-2R1+R2 \\ -2R1+R3}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & -1 & -4 \\ 0 & -5 & -1 & -6 \end{array}\right]$$

$$\underset{-2R2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 6 & 2 & 8 \\ 0 & -5 & -1 & -6 \end{array}\right]$$

$$\underset{R3+R2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -5 & -1 & -6 \end{array}\right]$$

$$\underset{\substack{-R2+R1 \\ 5R2+R3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

$$\underset{\frac{1}{4}R3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\underset{-R3+R2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

So, the solution is $x=1$, $y=1$ and $z=1$, which means that $(1,1,1)$ is the point of intersection of the three planes.

8. Use the augmented matrix to solve the following system

$$\begin{array}{ccccc} ax & + & y & = & 1 \\ x & + & y & = & b \end{array}$$

for any real values of a and b.

<u>**Solution**</u>**:**

We first form the augmented matrix:

$$\left[\begin{array}{cc|c} a & 1 & 1 \\ 1 & 1 & b \end{array}\right]$$

$$\underset{\text{R1} \longleftrightarrow \text{R2}}{\sim} \quad \left[\begin{array}{cc|c} 1 & 1 & b \\ a & 1 & 1 \end{array}\right]$$

$$\underset{-a\text{R1}+\text{R2}}{\sim} \quad \left[\begin{array}{cc|c} 1 & 1 & b \\ 0 & (1-a) & (1-ab) \end{array}\right].$$

We now need to explore some cases:

Case 1: $a \neq 1$:

Then, $1-a \neq 0$, so we find

$$(1-a)y = 1-ab \Longrightarrow y = \frac{1-ab}{1-a}.$$

Next, $x+y=b$

$$\Longrightarrow x = b-y = \frac{b(1-a)}{1-a} - \frac{1-ab}{1-a} = \frac{b-1}{1-a}.$$

In this case, the system has the unique solution

$$\left\{ \begin{array}{ccc} x & = & \frac{b-1}{1-a} \\ \\ y & = & \frac{1-ab}{1-a} \end{array} \right. .$$

Case 2: $a = 1$ and $b \neq 1$:

Then, the augmented matrix becomes

$$\left[\begin{array}{cc|c} 1 & 1 & b \\ 0 & 0 & (1-b) \end{array}\right],$$

where $1 - b \neq 0$. Therefore, in this case, the system is inconsistent.

Case 3: $a = 1$ and $b = 1$:

Now the augmented matrix becomes

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

In this case, the system is consistent. Because the reduced form of its augmented matrix only has one leading one, the system will have an arbitrary variable and thus infinitely many solutions. In particular, the general solution of the system is

$$\left\{\begin{array}{rcr} x & = & 1-y \\ \\ y & = & y \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{rcr} x & = & 1-t \\ \\ y & = & t \end{array}\right. .$$

You need Felynx Cougati's linear equation solver to do the next problem. (Go to http://www.ualberta.ca/dept/math/gauss/fcm/ and open linear algebra in $\mathcal{R}^n$. Click (here) and then Linear Equations and go to the method of Gauss Jordan elimination. There you will find "row reduced applet.")

9. Solve the following system by using Gauss Jordan elimination method:

$$\left\{\begin{array}{rcrcrcr} 1.1x & + & 2.1y & + & .9z & = & 4.1 \\ x & - & 1.1y & + & .1z & = & 0 \\ 1.4x & + & 2.7y & + & 3z & = & 7.1 \end{array}\right.$$

Solution:

The solution is

$$\left\{\begin{array}{rcl} x & = & 1 \\ y & = & 1 \\ z & = & 1 \end{array}\right. .$$

You can verify that this is the correct solution by substituting these values into the original system.

However, you were asked to use the software. Depending on the method that you have used, the values you have obtained are probably *close* but not exactly the same. This is due to the "roundoff error" that occurs with the use of any calculator. One method gives $x = 1.0000002$, $y = 1$ and $z = 0.9999999$.