Solutions for Assignment 2

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For each of the following matrices apply elementary row operations to find the reduced echelon form.

1. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

Solution:

$$\begin{aligned}
& \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \\
\underset{\substack{-\mathsf{R1}+\mathsf{R2} \\ -2\mathsf{R1}+\mathsf{R3}}}{\sim} \quad & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} \\
\underset{\mathsf{R2} \longleftrightarrow \mathsf{R3}}{\sim} \quad & \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \\
\underset{\substack{-\mathsf{R2} \\ -\mathsf{R3}}}{\sim} \quad & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
\underset{-\mathsf{R2}+\mathsf{R1}}{\sim} \quad & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
\underset{-\mathsf{R3}+\mathsf{R2}}{\sim} \quad & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

2. $$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\underset{\substack{R1+R2 \\ -R1+R3}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{-R2+R1}{\sim} \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

Solution:

$$\underset{\substack{-R1+R2 \\ -2R1+R3 \\ -3R1+R4}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\underset{-R2+R4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\underset{-R3+R4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4. $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \end{bmatrix}$

<u>Solution</u>:

$$\begin{array}{cc} & \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \end{bmatrix} \\ \begin{array}{c} \sim \\ -2\text{R1}+\text{R2} \\ -3\text{R1}+\text{R3} \end{array} & \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -2 & -4 & -6 \\ 0 & -4 & -8 & -12 \end{bmatrix} \\ \begin{array}{c} \sim \\ -\frac{1}{2}\text{R2} \end{array} & \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -4 & -8 & -12 \end{bmatrix} \\ \begin{array}{c} \sim \\ -3\text{R2}+\text{R1} \\ 4\text{R2}+\text{R3} \end{array} & \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{array}$$

Find the general solutions of the systems whose augmented matrices are given below:

5. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 1 & 1 & 0 \end{array}\right]$

<u>Solution</u>:

Notice that the question did not specify which variables were used in the original system. In this situation, you must specify the variables *and* their order.

We first note that there are four coefficient columns in the augmented matrix, so we need four variables. One option is w, x, y, z. Another option is to use subscripted variables, such as x_1, x_2, x_3, x_4. The benefit of the latter is that we can create any number of variables, whereas in the former we are limited by the number of letters in the English alphabet!

After the variables have been specified, there are two ways of solving the system.

Method 1: We can use *forward* substitution. For example, the first equation tells us that $x_1 = 0$. We substitute that into the second equation, $x_1 + x_2 = 0$, to find that $x_2 = 0$, and so on.

Method 2: *We can use the result from Question 3!* Note that a column of all zeros will not be changed by any elementary row operation, so the reduced echelon form of the augmented matrix above will be the reduced echelon form of its coefficient matrix (i.e. the answer to Question 3) augmented with a column of all zeros:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Using either method, the solution is

$$\left\{\begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & 0 \\ x_3 & = & 0 \\ x_4 & = & 0 \end{array}\right. .$$

6. $$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \end{array}\right]$$

<u>Solution</u>:

Similar to Method 2 above, we can use the result from Question 4 to find the reduced echelon form of the augmented matrix above:

$$\left[\begin{array}{ccr|r} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Then, there is no leading 1 in the coefficient column of x_3, so x_3 is arbitrary. Solving equations 1 and 2 for x_1 and x_2 respectively, we get $x_1 = -2 + x_3$ and $x_2 = 3 - 2x_3$. Hence, the general solution is

$$\left\{\begin{array}{rcr} x_1 & = & -2+t \\ x_2 & = & 3-2t \\ x_3 & = & t \end{array}\right. .$$

Solve the following systems of equations:

7. $$\begin{array}{rcrcrcrcr} x^2 & + & y^2 & - & z^2 & & & = & 1 \\ 3x^2 & - & 3y^2 & + & z^2 & & & = & 1 \\ 2x^2 & + & y^2 & - & z^2 & & & = & 2 \\ x & + & y & + & z & + & w & = & 10 \end{array}$$

Solution:

We first need to solve the system formed by the first three equations:

$$\left\{ \begin{array}{rcrcrcr} x^2 & + & y^2 & - & z^2 & = & 1 \\ 3x^2 & - & 3y^2 & + & z^2 & = & 1 \\ 2x^2 & + & y^2 & - & z^2 & = & 2 \end{array} \right. .$$

Note that this system is not linear! However, we can still use the same techniques for solving linear systems to solve this system: the trick is to consider x^2, y^2, and z^2 as the variables. Alternately, if we replace x^2, y^2, and z^2 by three new variables, then the resulting system will be linear in these new variables. For example, defining $p = x^2$, $q = y^2$, and $r = z^2$ yields

$$\left\{ \begin{array}{rcrcrcr} p & + & q & - & r & = & 1 \\ 3p & - & 3q & + & r & = & 1 \\ 2p & + & q & - & r & = & 2 \end{array} \right. .$$

Now we can find the reduced echelon form of the augmented matrix of this system:

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 3 & -3 & 1 & 1 \\ 2 & 1 & -1 & 2 \end{array} \right]$$

$$\underset{\substack{-3\mathsf{R1}+\mathsf{R2} \\ -2\mathsf{R1}+\mathsf{R3}}}{\sim} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 0 & -6 & 4 & -2 \\ 0 & -1 & 1 & 0 \end{array} \right]$$

$$\underset{\mathsf{R2} \longleftrightarrow \mathsf{R3}}{\sim} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -6 & 4 & -2 \end{array} \right]$$

$$\underset{-\mathsf{R2}}{\sim} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -6 & 4 & -2 \end{array} \right]$$

$$\underset{\substack{-R2+R1\\6R2+R3}}{\sim}\quad\left[\begin{array}{ccr|r}1&0&0&1\\0&1&-1&0\\0&0&-2&-2\end{array}\right]$$

$$\underset{-\frac{1}{2}R3}{\sim}\quad\left[\begin{array}{ccr|c}1&0&0&1\\0&1&-1&0\\0&0&1&1\end{array}\right]$$

$$\underset{R3+R2}{\sim}\quad\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\0&0&1&1\end{array}\right].$$

So, the solution to *this* system is

$$\begin{cases} p &= 1\\ q &= 1\\ r &= 1 \end{cases}.$$

Ergo, the *original* system is equivalent to

$$\begin{cases} x^2 & & & & & & & = & 1\\ & & y^2 & & & & & = & 1\\ & & & & z^2 & & & = & 1\\ x & + & y & + & z & + & w & = & 10 \end{cases}$$

$$\sim \begin{cases} x &= \pm 1\\ y &= \pm 1\\ z &= \pm 1\\ w &= 10-x-y-z. \end{cases}$$

Since there two choices for the value of x, two choices for the value of y, and two choices for the value of z, there are $2\cdot 2\cdot 2 = 8$ solutions. Each of these solutions is listed below in the form (x,y,z,w):

$$(1,1,1,7),\ (1,1,-1,9),\ (1,-1,1,9),\ (1,-1,-1,11),\ (-1,1,1,9),$$
$$(-1,1,-1,11),\ (-1,-1,1,11),\ \text{and}\ (-1,-1,-1,13).$$

8. $$\begin{array}{rcrcrcr} 3x & + & y & - & 4z & = & 0 \\ 2x & - & 3y & + & z & = & 0 \\ 7x & - & 5y & - & 2z & = & 0 \\ x^2 & + & y^2 & + & 2z^2 & = & 16 \end{array}$$

<u>Solution</u>:

The solutions of this system are the solutions of the linear system formed by the first three equations that also satisfy the fourth equation. So, we first solve the first three equations and then determine which of those solutions (if any) satisfy the fourth equation.

To begin, the system formed by the first three equations is

$$\left\{ \begin{array}{rcrcrcr} 3x & + & y & - & 4z & = & 0 \\ 2x & - & 3y & + & z & = & 0 \\ 7x & - & 5y & - & 2z & = & 0 \end{array} \right. .$$

We now find the reduced echelon form of the augmented matrix:

$$\left[\begin{array}{rrr|r} 3 & 1 & -4 & 0 \\ 2 & -3 & 1 & 0 \\ 7 & -5 & -2 & 0 \end{array} \right]$$

$$\underset{-\mathrm{R2}+\mathrm{R1}}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & -5 & 0 \\ 2 & -3 & 1 & 0 \\ 7 & -5 & -2 & 0 \end{array} \right]$$

$$\underset{\substack{-2\mathrm{R1}+\mathrm{R2} \\ -7\mathrm{R1}+\mathrm{R3}}}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & -5 & 0 \\ 0 & -11 & 11 & 0 \\ 0 & -33 & 33 & 0 \end{array} \right]$$

$$\underset{-\frac{1}{11}\mathrm{R2}}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & -5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -33 & 33 & 0 \end{array} \right]$$

$$\underset{\substack{-4\mathrm{R2}+\mathrm{R1} \\ 33\mathrm{R2}+\mathrm{R3}}}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

There is no leading 1 in the coefficient column of z, so z is arbitrary. Solving equations 1 and 2 for x and y respectively yields $x = z$ and $y = z$.

Therefore, the general solution of the system formed by the first three equations is

$$\begin{cases} x &= z \\ y &= z \\ z &= z \end{cases}.$$

Next, to determine which of these solutions satisfy the fourth equation, we substitute these values of x, y, and z into the fourth equation:

$$\begin{aligned} & x^2+y^2+2z^2=16 \\ \Longrightarrow\quad & (z)^2+(z)^2+2z^2=16 \\ \Longrightarrow\quad & 4z^2=16 \\ \Longrightarrow\quad & z^2=4 \\ \Longrightarrow\quad & z=\pm 2. \end{aligned}$$

Thus, there are two solutions,

$$\begin{cases} x &= 2 \\ y &= 2 \\ z &= 2 \end{cases} \quad \text{and} \quad \begin{cases} x &= -2 \\ y &= -2 \\ z &= -2 \end{cases}.$$