# Math 1410–Solutions for Assignment 4

Submitted Friday, October 14

1. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, find a matrix $B$ such that $BA$ is the the reduced echelon form of $A$.

**<u>Solution</u>:**

We need to find the reduced echelon form of the augmented matrix $[\,A \mid I\,]$. After we have done that, the matrix on the right side of the partition will be $B$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\substack{-2\mathsf{R1}+\mathsf{R2} \\ -\mathsf{R1}+\mathsf{R3}}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\underset{\mathsf{R2} \longleftrightarrow \mathsf{R3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\underset{-\mathsf{R2}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\underset{-\mathsf{R2}+\mathsf{R1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\underset{-\mathsf{R3}+\mathsf{R1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right].$$

Therefore, $B = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$.

2. Find the inverse matrix (if there is one) of each of the following matrices:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

**<u>Solution</u>:**

For each matrix $C$, we find a matrix $M$ such that $MC$ is the reduced echelon form of $C$. If $MC = I$, then $C^{-1} = M$; otherwise, $C$ has no inverse.

$$[\, A \mid I \,]$$

$$= \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\text{R1} \longleftrightarrow \text{R2}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\substack{-2\text{R1}+\text{R2} \\ -3\text{R1}+\text{R3}}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \end{array}\right]$$

$$\underset{-\text{R2}+\text{R3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right]$$

$$\underset{\frac{1}{2}\text{R3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\underset{\substack{-\text{R3}+\text{R1} \\ \text{R3}+\text{R2}}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right].$$

Thus, $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{5}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$

Next, $[\,B \mid I\,]$

$$= \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\substack{-\mathrm{R1}+\mathrm{R2} \\ \mathrm{R1}+\mathrm{R3}}}{\sim} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{-\mathrm{R2}+\mathrm{R4}}{\sim} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Because of the row of zeros on the left side of the partition, it is clear that the reduced echelon form of $B$ is not $I$. Hence, we can stop at this point* and say that $B$ has no inverse.

*Notice that the problem did not ask us to find the reduced echelon form of $B$ or the matrix $M$ such that $MB$ is the reduced echelon form of $B$.

3. Write the following system of equations in matrix form $AX = B$ and then use it to solve the system (note that the matrix $A$ is the same as in problem 2):

$$\begin{array}{rcrcrcl} 2x & + & y^2 & + & z^3 & = & 2 \\ x & & & + & z^3 & = & 0 \\ 3x & + & y^2 & + & 4z^3 & = & 0 \end{array}$$

**<u>Solution</u>:**

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y^2 \\ z^3 \end{bmatrix}$, and $B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

Then, the matrix form $AX = B$ of the system above is

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y^2 \\ z^3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Since $A^{-1}$ exists (we found it in Problem 2), we can premultiply (leftmultiply) both sides of the equation $AX = B$ to get

$$\begin{aligned} & A^{-1}(AX) = A^{-1}(B) \\ \Longrightarrow \quad & \left(A^{-1}A\right)X = A^{-1}B \\ \Longrightarrow \quad & IX = A^{-1}B \\ \Longrightarrow \quad & X = A^{-1}B. \end{aligned}$$

Consequently,

$$\begin{aligned} \begin{bmatrix} x \\ y^2 \\ z^3 \end{bmatrix} &= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{5}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

So, $x = 1$, $y^2 = 1$, and $z^3 = -1$, which means that $x = 1$, $y = \pm 1$, and $z = -1$. Written in the form $(x, y, z)$, the solutions of the system are

$$(1,1,1) \quad \text{and} \quad (1,-1,1).$$

4. Find $B^{-1}$ if

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 0 \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} -2 & 1 & -6 \\ 3 & 1 & 10 \\ 6 & 2 & -4 \end{bmatrix}.$$

**<u>Solution</u>:**

Note that $(AB)^{-1} = B^{-1}A^{-1}$, since $(AB)\left(B^{-1}A^{-1}\right) = \left(B^{-1}A^{-1}\right)(AB) = I$. Also note that $\left(B^{-1}A^{-1}\right)A = B^{-1}\left(A^{-1}A\right) = B^{-1}I = B^{-1}$.

Ergo,

$$\begin{aligned} B^{-1} &= (AB)^{-1}A \\ &= \begin{bmatrix} -2 & 1 & -6 \\ 3 & 1 & 10 \\ 6 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -14 & -1 & -5 \\ 23 & 5 & 0 \\ -2 & -14 & 0 \end{bmatrix}. \end{aligned}$$

5. Let $P$ be a matrix such that $PP^t = nI$, where $n$ is a nonzero number. Show that

$$P^{-1} = \frac{1}{n}P^t.$$

**<u>Solution</u>:**

We first need to assume that $P$ is a square matrix. Otherwise, the above statement is not true!

In particular, if $P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$, then $PP^t = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = 9I$, but $P$ is not invertible!

Assuming that $P$ is square, there are two ways of showing that $P^{-1} = \frac{1}{n}P^t$.

**Method 1:** Multiply $P$ and $\frac{1}{n}P^t$:

$$P\left(\frac{1}{n}P^t\right) = \frac{1}{n}\left(PP^t\right) = \frac{1}{n}(nI) = I.$$

Since $P$ and $\frac{1}{n}P^t$ are both square matrices, it follows that $\left(\frac{1}{n}P^t\right)P = I$ also. Hence, $\frac{1}{n}P^t$ is the inverse of $P$.

**Method 2:** Manipulate the equation $PP^t = nI$ into the form $PC = I$:

$$\begin{aligned} & PP^t = nI \\ \Longrightarrow\quad & \tfrac{1}{n}\left(PP^t\right) = \tfrac{1}{n}(nI) \\ \Longrightarrow\quad & P\left(\tfrac{1}{n}P^t\right) = I. \end{aligned}$$

Again, $P$ and $\frac{1}{n}P^t$ are both square, so $\left(\frac{1}{n}P^t\right)P = I$. Therefore, $P^{-1} = \frac{1}{n}P^t$.

Use this to find $P^{-1}$, if

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

To determine $n$, we find the product $PP^t$

$$\begin{aligned}
&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}^t \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \\
&= 4I.
\end{aligned}$$

Consequently, $n = 4$, so

$$P^{-1} = \frac{1}{4}P^t = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

6. (Bonus problem) $A$ is a $2 \times 2$ matrix. Show that if $AB = BA$ for all $2 \times 2$ matrices $B$, then $A = aI$ for some number $a$.

<u>**Solution**</u>**:**

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, there are two ways to show that $A = aI$ for some number $a$.

**Method 1:** We can substitute any $2 \times 2$ matrix in for $B$ in the equation $AB = BA$:

$$A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} A$$

$$\Longrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Longrightarrow \quad \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \quad b = 0 \text{ and } c = 0.$$

Also,

$$A \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} A$$

$$\Longrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Longrightarrow \quad \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \quad a = d.$$

Thus, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = aI$, as required.

**Method 2:** Let $B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$. Then,

$$
\begin{aligned}
& AB = BA \\
\Longrightarrow \quad & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
\Longrightarrow \quad & \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} = \begin{bmatrix} aw+cx & bw+dx \\ ay+cz & by+dz \end{bmatrix} \\
\Longrightarrow \quad & \left\{ \begin{array}{rcl} aw+by & = & aw+cx \\ ax+bz & = & bw+dx \\ cw+dy & = & ay+cz \\ cx+dz & = & by+dz \end{array} \right. \\
\Longrightarrow \quad & \left\{ \begin{array}{rcl} by & = & cx \\ ax+bz & = & bw+dx \\ cw+dy & = & ay+cz \\ cx & = & by \end{array} \right. .
\end{aligned}
$$

Since $B$ can be any $2 \times 2$ matrix, $w$, $x$, $y$, and $z$ can be any real numbers. If we make $x = 0$ and $y = 1$, we find that $b$ must be 0. If we make $x = 1$ and $y = 0$, we find that $c$ must be 0 as well.

As a result, the equation $cw + dy = ay + cz$ becomes $dy = ay$. If we make $y = 1$, we find that $d = a$. Hence,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = aI.$$