Math 1410–Solutions for Assignment 8

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1. Let $A = \begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 1 \\ -1 & 2 & 0 & 0 & 2 \end{bmatrix}$. Find the dimension of:

   (a) the row space of A,

   <u>Solution</u>:

   We begin by finding the reduced echelon form of A:

$$\begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 1 \\ -1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

$$\underset{\mathsf{R1+R3}}{\sim} \begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\underset{\substack{\mathsf{R2+R1} \\ \mathsf{-R2+R3}}}{\sim} \begin{bmatrix} \textcircled{1} & 0 & 2 & 2 & 0 \\ 0 & \textcircled{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   There are two leading ones, so any basis of the row space of A contains two vectors. Therefore, the dimension of the row space of A is 2.

   (b) the solution set of the equation $A\underline{x} = 0$ (note that $\underline{x}$ is a column vector).

   <u>Solution</u>:

   The augmented matrix is $\left[\begin{array}{ccccc|c} 1 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 0 & 2 & 0 \end{array}\right]$.

Using the work done in part (a), the reduced echelon form of this augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Using the variables $x,\ y,\ z,\ s,$ and $t,$ the solution to this system is

$$\begin{cases} x &= -2z-2s \\ y &= -z-s-t \\ z &= z \\ s &= s \\ t &= t. \end{cases}$$

Then, every solution has the form

$$\begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2z-2s \\ -z-s-t \\ z \\ s \\ t \end{bmatrix} = z \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, the set $\{(-2,-1,\ 1,\ 0,\ 0),\ (-2,-1,\ 0,\ 1,\ 0),\ (0,-1,\ 0,\ 0,\ 1)\}$ is a basis of the solution set of this system. Thus, the dimension of the solution set of $A\underline{x} = 0$ is 3.

2. Let $\underline{v} = (-2,\ 5,\ 2,\ 4)$ and $\underline{u} = (1,\ 1,\ 0,-1)$.

(a) Find the projection of $\underline{v}$ on $\underline{u} = (1,\ 1,\ 0,-1)$ and call it $\underline{w}$.

<u>Solution</u>:

$$\underline{w} = \text{proj}_{\underline{u}}\underline{v} = \frac{\underline{v} \circ \underline{u}}{\underline{u} \circ \underline{u}}\,\underline{u}$$

$$= \frac{(-2,\ 5,\ 2,\ 4) \circ (1,\ 1,\ 0,-1)}{(1,\ 1,\ 0,-1) \circ (1,\ 1,\ 0,-1)} (1,\ 1,\ 0,-1)$$

$$= \frac{-2+5+0-4}{1+1+0+1} (1,\ 1,\ 0,-1)$$

$$= \frac{-1}{3} (1,\ 1,\ 0,-1)$$

$$= \left(-\frac{1}{3}, -\frac{1}{3},\ 0,\ \frac{1}{3}\right).$$

(b) Find the vector $\underline{x} = \underline{v} - \underline{w}$.

<u>Solution</u>:

$$\underline{x} = (-2,\ 5,\ 2,\ 4) - \left(-\frac{1}{3}, -\frac{1}{3},\ 0,\ \frac{1}{3}\right) = \left(-\frac{5}{3},\ \frac{16}{3},\ 2,\ \frac{11}{3}\right).$$

(c) Find $\underline{u} \circ \underline{x}$.

<u>Solution</u>:

$$\underline{u} \circ \underline{x} = (1,\ 1,\ 0,-1) \circ \left(-\frac{5}{3},\ \frac{16}{3},\ 2,\ \frac{11}{3}\right) = -\frac{5}{3} + \frac{16}{3} - \frac{11}{3} = 0.$$

3. (a) Show that the two vectors $\underline{u} = (1,1,1)$ and $\underline{v} = (1,-2,1)$ are orthogonal.

<u>Solution</u>:

Two vectors are orthogonal if their dot product is zero, and

$$\underline{u} \circ \underline{v} = (1,\ 1,\ 1) \circ (1,-2,\ 1) = 1 - 2 + 1 = 0,$$

so $\underline{u}$ and $\underline{v}$ are orthogonal.

(b) Let $\underline{w} = (1,0,1)$. Find $\text{proj}_{\underline{u}}\underline{w}$ and $\text{proj}_{\underline{v}}\underline{w}$.

Solution:

$$\text{proj}_{\underline{u}}\underline{w} = \frac{\underline{w} \circ \underline{u}}{\underline{u} \circ \underline{u}}\,\underline{u} = \frac{(1,\ 0,\ 1) \circ (1,\ 1,\ 1)}{(1,\ 1,\ 1) \circ (1,\ 1,\ 1)}\,(1,\ 1,\ 1)$$
$$= \frac{1+0+1}{1+1+1}\,(1,\ 1,\ 1) = \frac{2}{3}(1,\ 1,\ 1) = \left(\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3}\right).$$

Next,

$$\text{proj}_{\underline{v}}\underline{w} = \frac{\underline{w} \circ \underline{v}}{\underline{v} \circ \underline{v}}\,\underline{v} = \frac{(1,\ 0,\ 1) \circ (1,-2,\ 1)}{(1,-2,\ 1) \circ (1,-2,\ 1)}\,(1,-2,\ 1)$$
$$= \frac{1+0+1}{1+4+1}\,(1,-2,\ 1) = \frac{2}{6}(1,-2,\ 1) = \left(\frac{1}{3},-\frac{2}{3},\ \frac{1}{3}\right).$$

(c) Verify that $\underline{w} = \text{proj}_{\underline{u}}\underline{w} + \text{proj}_{\underline{v}}\underline{w}$.

Solution:

$$\text{proj}_{\underline{u}}\underline{w} + \text{proj}_{\underline{v}}\underline{w} = \left(\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3}\right) + \left(\frac{1}{3},-\frac{2}{3},\ \frac{1}{3}\right)$$
$$= \left(\frac{3}{3},\ \frac{0}{3},\ \frac{3}{3}\right) = (1,\ 0,\ 1) = \underline{w}.$$

4. (a) Verify that the set of vectors

$$S = \{(1,\ 1,\ 1,\ 1),\ (1,-1,\ 1,-1),\ (1,-1,-1,\ 1),\ (1,\ 1,-1,-1)\}$$

forms an orthogonal basis for $\mathbb{R}^4$.

Solution:

Let $\underline{v}_1 = (1,\ 1,\ 1,\ 1)$, $\underline{v}_2 = (1,-1,\ 1,-1)$, $\underline{v}_3 = (1,-1,-1,\ 1)$, and $\underline{v}_4 = (1,\ 1,-1,-1)$, so that $S = \{\underline{v}_1,\ \underline{v}_2,\ \underline{v}_3,\ \underline{v}_4\}$.

Then,

$$\begin{aligned}
\underline{v}_1 \circ \underline{v}_2 &= 1-1+1-1 = 0,\\
\underline{v}_1 \circ \underline{v}_3 &= 1-1-1+1 = 0,\\
\underline{v}_1 \circ \underline{v}_4 &= 1+1-1-1 = 0,\\
\underline{v}_2 \circ \underline{v}_3 &= 1+1-1-1 = 0,\\
\underline{v}_2 \circ \underline{v}_4 &= 1-1-1+1 = 0,\\
\text{and } \underline{v}_3 \circ \underline{v}_4 &= 1-1+1-1 = 0,
\end{aligned}$$

so S is an orthogonal set.

A nonempty orthogonal set of nonzero vectors is linearly independent, so S is linearly independent. Consequently, S is a basis of its span. Since S contains 4 vectors, the dimension of the span of S is 4. In other words, the span of S is all of $\mathbb{R}^4$.

Hence, S is an orthogonal basis of $\mathbb{R}^4$.

(b) Use part (a) to express the vector $(1,2,3,4)$ as a linear combination of the vectors in S.

<u>Solution</u>:

Let $\underline{u} = (1,\ 2,\ 3,\ 4)$. Since S is orthogonal,

$$\begin{aligned}
\underline{u} &= \text{proj}_{\underline{v}_1}\underline{u} + \text{proj}_{\underline{v}_2}\underline{u} + \text{proj}_{\underline{v}_3}\underline{u} + \text{proj}_{\underline{v}_4}\underline{u}\\
&= \frac{\underline{u} \circ \underline{v}_1}{\underline{v}_1 \circ \underline{v}_1}\,\underline{v}_1 + \frac{\underline{u} \circ \underline{v}_2}{\underline{v}_2 \circ \underline{v}_2}\,\underline{v}_2 + \frac{\underline{u} \circ \underline{v}_3}{\underline{v}_3 \circ \underline{v}_3}\,\underline{v}_3 + \frac{\underline{u} \circ \underline{v}_4}{\underline{v}_4 \circ \underline{v}_4}\,\underline{v}_4\\
&= \frac{(1,\ 2,\ 3,\ 4) \circ (1,\ 1,\ 1,\ 1)}{(1,\ 1,\ 1,\ 1) \circ (1,\ 1,\ 1,\ 1)}\,\underline{v}_1 + \frac{(1,\ 2,\ 3,\ 4) \circ (1,-1,\ 1,-1)}{(1,-1,\ 1,-1) \circ (1,-1,\ 1,-1)}\,\underline{v}_2\\
&\quad + \frac{(1,\ 2,\ 3,\ 4) \circ (1,-1,-1,\ 1)}{(1,-1,-1,\ 1) \circ (1,-1,-1,\ 1)}\,\underline{v}_3 + \frac{(1,\ 2,\ 3,\ 4) \circ (1,\ 1,-1,-1)}{(1,\ 1,-1,-1) \circ (1,\ 1,-1,-1)}\,\underline{v}_4
\end{aligned}$$

$$= \tfrac{1+2+3+4}{1+1+1+1}\,\underline{v}_1 + \tfrac{1-2+3-4}{1+1+1+1}\,\underline{v}_2 + \tfrac{1-2-3+4}{1+1+1+1}\,\underline{v}_3 + \tfrac{1+2-3-4}{1+1+1+1}\,\underline{v}_4$$

$$= \tfrac{10}{4}\,\underline{v}_1 + \tfrac{-2}{4}\,\underline{v}_2 + \tfrac{0}{4}\,\underline{v}_3 + \tfrac{-4}{4}\,\underline{v}_4$$

$$= \tfrac{5}{2}\,\underline{v}_1 - \tfrac{1}{2}\,\underline{v}_2 + 0\,\underline{v}_3 - 1\,\underline{v}_4.$$

5. Let $\underline{a} = (-3,2,1)$, $\underline{b} = (1,1,1)$, and $\underline{c} = (9,-4,7)$ be vectors in $\mathbb{R}^3$.

(a) Show that vector $\underline{a}$ is orthogonal to the vector $\underline{b}$.

Solution:

$$\underline{a} \circ \underline{b} = (-3,\ 2,\ 1) \circ (1,\ 1,\ 1) = -3 + 2 + 1 = 0,$$

so the vectors $\underline{a}$ and $\underline{b}$ are orthogonal.

(b) Let $\underline{u} = \mathrm{proj}_{\underline{a}}\underline{c}$ and $\underline{v} = \mathrm{proj}_{\underline{b}}\underline{c}$. Find the vector $\underline{w} = \underline{c} - \underline{u} - \underline{v}$.

Solution:

$$\underline{u} = \frac{\underline{c} \circ \underline{a}}{\underline{a} \circ \underline{a}}\,\underline{a} = \frac{(9,-4,\ 7) \circ (-3,\ 2,\ 1)}{(-3,\ 2,\ 1) \circ (-3,\ 2,\ 1)}\,(-3,\ 2,\ 1)$$

$$= \frac{-27-8+7}{9+4+1}\,(-3,\ 2,\ 1) = \frac{-28}{14}\,(-3,\ 2,\ 1)$$

$$= -2\,(-3,\ 2,\ 1) = (6,-4,-2).$$

Next,

$$\underline{v} = \frac{\underline{c} \circ \underline{b}}{\underline{b} \circ \underline{b}}\,\underline{b} = \frac{(9,-4,\ 7) \circ (1,\ 1,\ 1)}{(1,\ 1,\ 1) \circ (1,\ 1,\ 1)}\,(1,\ 1,\ 1)$$

$$= \frac{9-4+7}{1+1+1}(1,\ 1,\ 1) = \frac{12}{3}(1,\ 1,\ 1)$$

$$= 4\,(1,\ 1,\ 1) = (4,\ 4,\ 4).$$

Finally,

$$\underline{w} = \underline{c} - \underline{u} - \underline{v} = (9, -4,\ 7) - (6, -4, -2) - (4,\ 4,\ 4)$$

$$= (\,9-6-4,\ -4+4-4,\ 7+2-4\,) = (-1, -4,\ 5).$$

(c) Show that vector $\underline{w}$ is orthogonal to both vectors $\underline{a}$ and $\underline{b}$.

<u>Solution</u>:

$$\underline{w} \circ \underline{a} = (-1, -4,\ 5) \circ (-3,\ 2,\ 1) = 3 - 8 + 5 = 0,$$

so $\underline{w}$ is orthogonal to $\underline{a}$. Next,

$$\underline{w} \circ \underline{b} = (-1, -4,\ 5) \circ (1,\ 1,\ 1) = -1 - 4 + 5 = 0,$$

so $\underline{w}$ is also orthogonal to $\underline{b}$.