TURYN-TYPE SEQUENCES: CLASSIFICATION, ENUMERATION AND CONSTRUCTION

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ABSTRACT. Turyn-type sequences, $TT(n)$, are quadruples of $\{\pm 1\}$-sequences $(A; B; C; D)$, with lengths $n, n, n, n-1$ respectively, where the sum of the nonperiodic autocorrelation functions of A, B and twice that of C, D is a δ-function (i.e., vanishes everywhere except at 0). Turyn-type sequences $TT(n)$ are known to exist for all even n not larger than 36. We introduce a definition of equivalence to construct a canonical form for $TT(n)$ in general. By using this canonical form, we enumerate the equivalence classes of $TT(n)$ for $n \leq 32$. We also construct the first example of Turyn-type sequences $TT(38)$.

2000 Mathematics Subject Classification 05B20, 05B30

1. Introduction

Let a *binary sequence* be a sequence $A = a_1, ..., a_m$ whose terms belong to $\{\pm 1\}$. To such a sequence, we associate the polynomial $A(x) = a_1 + a_2 x + \cdots + a_m x^{m-1}$, and refer to the Laurent polynomial $N(A) = A(x)A(x^{-1})$ as the *norm* of A. Denoted $TT(n)$, a *Turyn-type sequence* $(A; B; C; D)$ is a quadruple of binary sequences with A, B and C of length n and D of length $n-1$, such that

$$N(A) + N(B) + 2N(C) + 2N(D) = 6n - 2. \tag{1.1}$$

Turyn-type sequences should not be confused with the so called "Turyn Sequences" [7, Definition 5.1, p. 478], which are also quadruples of $\{\pm 1\}$-sequences but now of lengths $n, n, n-1, n-1$. In addition to the requirement that the sum of their non-periodic autocorrelation functions is a δ-function, they also have certain desirable symmetry properties. Unfortunately, there are only a few known Turyn Sequences, all with $n \leq 14$.

Turyn-type sequences play an important role in the construction of Hadamard matrices [4, 7]. For instance, the discovery of a Hadamard matrix of order 428 [5] used a $TT(36)$, constructed specifically for that

Key words and phrases. Turyn-type sequences, nonperiodic autocorrelation functions, canonical form.

purpose. From $TT(n)$, one can construct (as explained in Section 5) base sequences of lengths $2n-1$, $2n-1$, n, n. If base sequences of lengths m, m, n, n are known, one can use the Goethals-Seidel array to construct a Hadamard matrix of order $4(m+n)$. We refer the reader to [5, p. 436] for details.

Furthermore, two of the three remaining orders less than 1000 for which the existence of a Hadamard matrix is not known may be resolved by using Turyn-type sequences of appropriate lengths (assuming that they exist). More precisely, Turyn-type sequences $TT(56)$ and $TT(60)$ may be used to construct Hadamard matrices of orders 668 and 716 respectively.

The discovery of any new Turyn-type sequences leads to an infinite class of Hadamard matrices, as explained in [5, p. 439]. Despite the importance of Turyn-type sequences, not much is known about their existence. All the existing results related to these sequences rely on increasingly lengthy computer calculations. In order to have a better understanding of the structure of Turyn-type sequences, it is essential to classify them for as many values of n as possible. Our main goal is to provide a classification of $TT(n)$ for even $n \leq 32$ ($TT(n)$ do not exist for odd $n > 1$) and to modify an existing search method to construct a $TT(38)$. The new $TT(38)$ can be used to construct an infinite class of Hadamard matrices; see [5, p. 439].

In Section 2, we define the standard elementary transformations of $TT(n)$ and use them to introduce an equivalence relation. We also introduce a canonical form for Turyn-type sequences. Using this, we are able to compute the representatives of the equivalence classes.

An abstract group of order 2^{10} is introduced in Section 3, which acts naturally on all sets of $TT(n)$. The orbits of this group are the equivalence classes of $\{TT(n)\}$.

In Section 4, a list of representatives of the equivalence classes of $\{TT(n)\}$ (those for even $n \leq 32$) are tabulated. Due to their excessive length, the tables for $n > 10$ are truncated to 12 members only.

Finally, in Section 5, the search method for finding a $TT(38)$ is explained.

2. A canonical form for Turyn-type sequences

We denote finite sequences of integers by capital letters. If A is such a sequence of length n, then we denote its elements by the corresponding lower case letters. Thus,

$$A = a_1, a_2, \ldots, a_n.$$

The *nonperiodic autocorrelation function* of A, N_A, is defined by:

$$N_A(i) = \sum_{j \in \mathbb{Z}} a_j a_{i+j}, \quad i \in \mathbb{Z},$$

where $a_k = 0$ for $k < 1$ and $k > n$. (As usual, $\mathbb{Z}$ denotes the ring of integers.) Note that $N_A(-i) = N_A(i)$ for all $i \in \mathbb{Z}$ and $N_A(i) = 0$ for $i \geq n$. The integers $N_A(i)$ are the coefficients of the norm of A, i.e., we have

$$N(A) = \sum_{i \in \mathbb{Z}} N_A(i) x^i.$$

Assume that $(A; B; C; D)$ is a $TT(n)$. From equation (1.1), we have

$$N_A(i) + N_B(i) + 2N_C(i) + 2N_D(i) = 0, \quad i \neq 0. \tag{2.1}$$

The *negated* sequence, $-A$, the *reversed* sequence, A', and the *alternated* sequence, A^*, of the sequence A are defined by

$$\begin{aligned} -A &= -a_1, -a_2, \ldots, -a_n, \\ A' &= a_n, a_{n-1}, \ldots, a_1, \\ A^* &= a_1, -a_2, a_3, -a_4, \ldots, (-1)^{n-1} a_n \end{aligned}$$

respectively. Observe that $N(-A) = N(A') = N(A)$ and $N_{A^*}(i) = (-1)^i N_A(i)$ for all $i \in \mathbb{Z}$.

We define four types of elementary transformations of Turyn-type sequences.

The *elementary transformations* of $(A; B; C; D) \in \{TT(n)\}$ are the following:

(T1) Negate one of A, B, C or D.
(T2) Reverse one of A, B, C or D.
(T3) Alternate all four sequences A, B, C and D.
(T4) Interchange the sequences A and B.

We say that two $TT(n)$ are *equivalent* if one can be transformed to the other by applying a finite sequence of elementary transformations. One can enumerate the equivalence classes by finding suitable representatives of the classes. For that purpose, we introduce a canonical form.

Definition 2.1. We say that $S = (A; B; C; D) \in \{TT(n)\}$ is in *canonical form* if the following six conditions hold:

(i) $a_1 = a_n = b_1 = b_n = c_1 = d_1 = +1$;
(ii) If i is the least index such that $a_i \neq a_{n+1-i}$, then $a_i = +1$;
(iii) If i is the least index such that $b_i \neq b_{n+1-i}$, then $b_i = +1$;
(iv) If i is the least index such that $c_i = c_{n+1-i}$, then $c_i = +1$;
(v) If i is the least index such that $d_i d_{n-i} \neq d_{n-1}$, then $d_i = +1$;

(vi) Assume that $n > 2$. If $a_2 \neq b_2$ then $a_2 = +1$ and otherwise, $a_{n-1} = +1$ and $b_{n-1} = -1$.

Note that if $n > 1$, then (i) and (2.1) imply that $c_n = -1$. We can now prove that each equivalence class has a member which is in the canonical form. The uniqueness of this member will be proved in the next section.

Proposition 2.2. *Each equivalence class $\mathcal{E} \subseteq \{TT(n)\}$ has at least one member having the canonical form.*

Proof. Let $S = (A; B; C; D) \in \mathcal{E}$ be arbitrary. By applying the first three types of elementary transformations, we can assume that (i) holds. To satisfy the condition (ii), replace A with A' (if necessary), and similarly, we can satisfy the condition (iii). To satisfy the condition (iv), replace C with $-C'$ (if necessary).

To satisfy (v), observe that if i exists, then D is not symmetric and $1 < i \leq n/2$. If $d_{n-1} = +1$, it suffices to replace D with D' (if necessary). Otherwise, we replace D with $-D'$ (if necessary).

To satisfy (vi), observe that the condition (2.1) with $i = n-2$ implies that exactly one of the equalities $a_2 = b_2$ and $a_{n-1} = b_{n-1}$ holds. Thus, it suffices to apply T4 (if necessary). Hence, S is now in the canonical form. □

3. A symmetry group of $\{TT(n)\}$

We shall construct a group G of order 2^{10} which acts naturally on all $\{TT(n)\}$. Our (redundant) generating set for G will consist of 10 involutions. Each of these generators is an elementary transformation, and we use this information to construct G, i.e., to impose the defining relations. Let $S = (A; B; C; D)$ be an arbitrary member of $\{TT(n)\}$.

To construct G, we start with an elementary abelian group E of order 2^8 with generators ν_i, ρ_i, $i \in \{1, 2, 3, 4\}$. It acts on $\{TT(n)\}$ as follows:

$$\begin{aligned}
\nu_1 S &= (-A; B; C; D), & \rho_1 S &= (A'; B; C; D),\\
\nu_2 S &= (A; -B; C; D), & \rho_2 S &= (A; B'; C; D),\\
\nu_3 S &= (A; B; -C; D), & \rho_3 S &= (A; B; C'; D),\\
\nu_4 S &= (A; B; C; -D), & \rho_4 S &= (A; B; C; D').
\end{aligned}$$

That is, ν_i negates the ith sequence of S and ρ_i reverses it.

Next we introduce the involutory generator σ. We declare that σ commutes with $\nu_3, \nu_4, \rho_3, \rho_4$, and that

$$\sigma\nu_1 = \nu_2\sigma, \quad \sigma\rho_1 = \rho_2\sigma.$$

The group $H = \langle E, \sigma \rangle$ is the direct product of the group $H_1 = \langle \nu_1, \rho_1, \sigma \rangle$ of order 32 and $H_2 = \langle \nu_3, \nu_4, \rho_3, \rho_4 \rangle$. The action of E on $\{TT(n)\}$ extends to H by defining $\sigma S = (B; A; C; D)$.

Finally, we define G as the semidirect product of H and the group of order 2 with generator α. By definition, α satisfies $\alpha\rho_i\alpha = \rho_i\nu_i$ for $i = 1, 2, 3$ and commutes with ρ_4, σ and each ν_i. The action of H on $\{TT(n)\}$ extends to G by letting α act as the elementary transformation (T3), i.e., we have

$$\alpha S = (A^*; B^*; C^*; D^*).$$

We point out that the definition of G is independent of n.

The following proposition follows immediately from the construction of G and the description of its action on $\{TT(n)\}$.

Proposition 3.1. *The orbits of G in $\{TT(n)\}$ are the same as the equivalence classes.*

We shall need the following lemma.

Lemma 3.2. *For $S = (A; B; C; D) \in \{TT(n)\}$, set $\varphi(S) = a_1 a_n$. Then we have $\varphi(\alpha S) = -\varphi(S)$ and $\varphi(hS) = \varphi(S)$ for all $h \in H$.*

Proof. The first assertion holds because n is even. To prove the second assertion, it suffices to verify that it holds when h is one of the generators ν_j, ρ_j, $j = 1, 2, 3, 4$, or σ. This is obvious in the former case. It is also true in the latter case ($h = \sigma$) because equation (2.1) with $i = n - 1$ implies that $a_1 a_n = b_1 b_n$. □

The main tool that we use to enumerate the equivalence classes of $\{TT(n)\}$ is the following theorem.

Theorem 3.3. *For each equivalence class $\mathcal{E} \subseteq \{TT(n)\}$ there is a unique $S = (A; B; C; D) \in \mathcal{E}$ having the canonical form.*

Proof. In view of Proposition 2.2, we just have to prove the uniqueness assertion. Let

$$S^{(k)} = (A^{(k)}; B^{(k)}; C^{(k)}; D^{(k)}) \in \mathcal{E}, \quad (k = 1, 2)$$

be in the canonical form. We have to prove that in fact $S^{(1)} = S^{(2)}$.

By Proposition 3.1, we have $gS^{(1)} = S^{(2)}$ for some $g \in G$. We can write g as $g = \alpha^t h$ where $t \in \{0, 1\}$ and $h \in H$. The symbols (i)-(vi) will refer to the corresponding conditions of Definition 2.1.

Since both $S^{(1)}$ and $S^{(2)}$ have the canonical form, the condition (i) implies that $\varphi(S^{(1)}) = \varphi(S^{(2)}) = 1$, where φ is the function defined in Lemma 3.2. Now this lemma implies that $t = 0$, i.e., $g = h \in H$.

Recall that $H = H_1 \times H_2$. Thus, $g = h = h_1 h_2$ with $h_1 \in H_1$ and $h_2 \in H_2$. Consequently, $h_2 C^{(1)} = C^{(2)}$ and $h_2 D^{(1)} = D^{(2)}$. What we really mean by these equations is that we have

$$h_2 S^{(1)} = (A^{(1)}; B^{(1)}; C^{(2)}; D^{(2)}).$$

Hopefully this simplified notation for the action of H_2, as well as its analog for the action of H_1, will not lead to any confusion. We can write $h_2 = \nu_3^p \rho_3^q \nu_4^r \rho_4^s$ for some $p, q, r, s \in \{0, 1\}$. Then we have $\nu_3^p \rho_3^q C^{(1)} = C^{(2)}$ and $\nu_4^r \rho_4^s D^{(1)} = D^{(2)}$. We shall now prove that $C^{(1)} = C^{(2)}$ and $D^{(1)} = D^{(2)}$.

Since $c_1^{(1)} = c_1^{(2)} = 1$ and $c_n^{(1)} = c_n^{(2)} = -1$, we conclude that $p = q$. Now the condition (iv) implies that either $p = q = 0$ or $\nu_3 \rho_3 C^{(1)} = C^{(1)}$. In both cases we have $C^{(1)} = C^{(2)}$.

Since $d_1^{(1)} = d_1^{(2)} = 1$, the equality $\nu_4^r \rho_4^s D^{(1)} = D^{(2)}$ implies that $d_1^{(1)} d_{n-1}^{(1)} = d_1^{(2)} d_{n-1}^{(2)}$. Hence $d_{n-1}^{(1)} = d_{n-1}^{(2)} = \varepsilon$. If $\varepsilon = +1$ we must have $r = 0$ and the condition (v) shows that either $s = 0$ or $\rho_4 D^{(1)} = D^{(1)}$. In both cases, $D^{(1)} = D^{(2)}$. By a similar argument as in the previous paragraph but using the condition (v) instead of (iv), we can show that this equality also holds when $\varepsilon = -1$.

It remains to prove that $A^{(1)} = A^{(2)}$ and $B^{(1)} = B^{(2)}$. Since $a_1^{(1)} = a_n^{(1)} = a_1^{(2)} = a_n^{(2)} = +1$, we must have $h_1 \in \langle \rho_1, \rho_2, \sigma \rangle$, i.e., $h_1 = \rho_1^u \rho_2^v \sigma^w$ for some $u, v, w \in \{0, 1\}$. We claim that we can assume, without any loss of generality, that $w = 0$. This is clear if $A^{(1)} = B^{(1)}$. Otherwise, we have $n > 2$ and the condition (vi) implies that either $a_{n-1}^{(1)} = +1$, $b_{n-1}^{(1)} = -1$ and $a_{n-1}^{(1)} = b_{n-1}^{(1)}$ or $a_{n-1}^{(1)} = +1$, $b_{n-1}^{(1)} = -1$ and $a_1^{(1)} = b_1^{(1)}$. It is now easy to see that we must have $w = 0$. This proves our claim, and so we may assume that $h_1 = \rho_1^u \rho_2^v$. Consequently, we have $\rho_1^u A^{(1)} = A^{(2)}$ and $\rho_2^v B^{(1)} = B^{(2)}$. The condition (ii) implies that either $u = 0$ or $\rho_1 A^{(1)} = A^{(1)}$. In both cases we have $A^{(1)} = A^{(2)}$. The proof of $B^{(1)} = B^{(2)}$ is similar, using (iii) instead of (ii). □

4. Representatives of the equivalence classes

We have computed a set of representatives for the equivalence classes of Turyn-type sequences for even $n \leq 32$. Due to their excessive size, we tabulate whole sets for only $n \leq 12$. Each representative is given in the canonical form, which is made compact by using the following standard encoding scheme for Turyn-type sequences.

Let $S = (A; B; C; D) \in \{TT(n)\}$. For each index $i = 1, 2, \ldots, n-1$ the number $4(1-a_i)+2(1-b_i)+(1-c_i)+(1-d_i)/2$ is an integer in the range $0, 1, \ldots, 15$. We shall replace this integer by the corresponding hexadecimal digit $h_i \in \{0, 1, \ldots, 9, a, b, c, d, e, f\}$. We encode S by the sequence $h_1, h_2, \ldots, h_n$ of n hexadecimal digits. The hexadecimal digit h_n represents the number $2(1-a_n)+(1-b_n)+(1-c_n)/2 \in \{0, 1, \ldots, 7\}$.

Equivalently, if we apply the substitution $+1 \to 0$, $-1 \to 1$ to the sequence a_i, b_i, c_i, d_i for $i < n$, and the sequence a_n, b_n, c_n for $i = n$, then we obtain the binary representation of the hexadecimal digit h_i. Clearly, the encoding sequence $h_1, h_2, \ldots, h_n$ of S determines S uniquely.

As an example, the Turyn-type sequence

$$
\begin{aligned}
A &= \texttt{++-+-+-+}; \\
B &= \texttt{+------+}; \\
C &= \texttt{+--++++-}; \\
D &= \texttt{+++-++-}
\end{aligned}
$$

is encoded as $06e5c4d1$. Note that when displaying a binary sequence, we shall often write $+$ for $+1$ and $-$ for -1.

For each n, the representatives are listed in the lexicographic order of the symbol sequences $h_1, h_2, \ldots, h_n$. Since all representatives have the canonical form, we always have $h_1 = 0$ and $h_n = 1$. In tables 2-3, the last hexadecimal digit $h_n = 1$ is omitted. However, the first hexadecimal digit $h_1 = 0$ will always be recorded.

For $n \leq 10$ we list all representatives in Table 2. For $12 \leq n \leq 32$, we list in each case only the first dozen representatives. For $n \leq 22$, the list of representatives was computed independently by two different programs, but for the range $24 \leq n \leq 32$, only the more optimized program was used. We discuss the search method in the next section.

Table 1: The number of equivalence classes in $\{TT(n)\}$

n	2	4	6	8	10	12	14	16
# Eq. cl.	1	1	4	6	43	127	186	739
n	18	20	22	24	26	28	30	32
# Eq. cl.	675	913	3105	3523	3753	4161	4500	6226

Table 2: Class representatives for $n = 2, 4, 6, 8, 10$

			$n=2$				
1	0						
			$n=4$				
1	016						
			$n=6$				
1	006d6	2	01396	3	045ec	4	0608d
			$n=8$				
1	001c6a5	2	0049e25	3	005e5c6	4	00c1786
5	06e054d	6	06e5c4d				
			$n=10$				
1	0001f4a96	2	00036c796	3	0006f8365	4	000ef86a5
5	00134e696	6	001ce8965	7	0047e4f16	8	0049a13c6
9	0057c6e16	10	0076f4ee5	11	007809cd6	12	007b393e5
13	007cc94d6	14	007cca8e5	15	00870bec6	16	008f4dac6
17	00b6fa2e5	18	00c5c7e85	19	00e063895	20	00f6e8ea5
21	012408f96	22	01402b8e5	23	014308ae5	24	0401368bc
25	044a18fec	26	04932a63c	27	05176df5c	28	052bb137c
29	05716d9dc	30	0588caf1c	31	05a82aedc	32	05b7b13dc
33	05bf1b5dc	34	05fb71f5c	35	061137b4d	36	06113b58d
37	0614aec8d	38	061ae6e8d	39	061b3738d	40	061d7f54d
41	06a1058cd	42	06bcd84cd	43	074625ccd		

Table 3: First twelve class representatives for $n = 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32$

	$n = 12$		
1	0004f90bc96	2	0006b8c1da5
3	0007c918e96	4	0008bd43c96
5	0009e0a7c95	6	000b0f68d66
7	000b8d50e96	8	000d26db4a6
9	000d2e974a6	10	000d2e978a6
11	000e471ea96	12	000f0736695
	$n = 14$		
1	00036ac71c765	2	00041f906bca5
3	000497813eca5	4	0006698fc23a5
5	0007b2af4e3a5	6	0007b2b343e95
7	0008e783d62a5	8	000a07d41ad96
9	000af2175a396	10	000b31c7563a5
11	000b6283acd65	12	000b679e32ea5
	$n = 16$		
1	0000778e52de556	2	00007e4b0e53956
3	0000f0d734a5966	4	0000f5461f2a965
5	0000fdc397459a5	6	0000fdc397499a5
7	00018f07d45ea95	8	0001c39c6e95965
9	00023e1c6748795	10	00023e1c6b48755
11	00049b15e4d3ca5	12	0004fe172b471a6
	$n = 18$		
1	00006758b30d1e9a5	2	0000b7c117952e9a5
3	0000f87341bd29956	4	000149f0b259ee595
5	00017c2183a68f655	6	0001897a4c3df0596
7	0001b465432e0fa95	8	0001cb44731d2a9a5
9	00030e9bb21da8b65	10	000363645f0e52b95
11	000366969e231c755	12	0004b350d6918f1a6
	$n = 20$		
1	000038e2739c7a0b695	2	00004ef0b7c0b6bc5a6
3	00006bcab161e913a65	4	000077078d6f2433a95
5	0000bb0754e3e523695	6	0000bf40b3a3938d696
7	0000cb30fe68a5f86a5	8	0000e8af34cb43e95a6
9	0000f04b72a1f196a65	10	0000f0b27acca39a695
11	0000f0f216c9ba59aa6	12	0000f0f4ce7a15aa966
	$n = 22$		
1	00000f702c71a9ad56596	2	00000f7a12bd68e36a596
3	00000f8b358d263c5aa56	4	00000fb60539ea1ea69a5
5	000032f0f792e9665b966	6	000032f87f835e2a57966
7	00003609cf34a6d81b9a5	8	0000376a43258e2dcb965
9	00003fa2c8bc24bd47a56	10	00004f1acf9149e8b2596
11	0000538b4dea91227c696	12	000058782f506bd31c966

		$n=24$	
1	00000b7c2cb2bc4b6cd9a96	2	00000dfc0c3f86787589a56
3	00000dfc0c3f8a747985a96	4	00000f9e90729c9f4ca55a6
5	00000fb24bcf48d26e55a56	6	00002671f06b3c7a41d7a96
7	00003ba5d1f0b55ac1c7956	8	000044bb0787c2d92ed1596
9	00004996a5f086ef03dc965	10	00004b67a135ca713cf2a56
11	00004f2b6038d5ac19bc695	12	00004fe0fdc0a7a498b1695
		$n=26$	
1	000000ff0f846f1ca5a5aa955	2	00000b70c5f25257c69c39966
3	00000b70cb54b0f1ea6239965	4	00000bab68f0da58e311d6a95
5	00000c7e12e4391b865f8a596	6	00000f8f50cb26da9e51a9996
7	00001477c0bed592960f39a55	8	000014bcf58a5f11269f05966
9	0000178b0f2d9285badc19a66	10	000017ac6234e90b6d7d25966
11	00002372d8f4a1ead7827b966	12	000027696c2491f88d3e0ba55
		$n=28$	
1	0000067cde3e50639ab46135aa5	2	000007f4038fa4d1529b16da656
3	00000ab877e0a8fd862df0396a5	4	00000b344e59ca17f29216e5695
5	00000df479ad14dab0c1f986a56	6	0000137872534b30ae5c2f69996
7	000013847ef03e69586e2e96596	8	000015c86f122d54bb8fc4da5a5
9	0000190ffe11a35f8695b709a96	10	00002799e66d6c8ebc25cf07aa5
11	00003065e3788a2e1d693e4b556	12	00003a6b92877521ef412d1b956
		$n=30$	
1	000000f70b106f9d427a25e9a9695	2	000003f0ed871781d5d2a65876956
3	000003f403872d2ba6cd5b1876a96	4	0000065f298b853ac3c2d86e39566
5	000007e6883ca99f22570f0ae55a5	6	000007f701bd8f28b1a2583ae9a56
7	00000bf4a07ab28c7dcd63e8da696	8	00000e3a785942359c33e0f669aa5
9	00000f0f1c3a662b3dc6a59669aa5	10	00000fb507b6a1c5b03ec70e69aa6
11	00000fe87624da3ac70bdeda59a66	12	00000ff118f947513c26d8a565a56
		$n=32$	
1	00000138f64f1c1e77844f26d95a596	2	0000067c7a5e84b6c1deb0cd71eaa65
3	000006d074e9e0fb056835f289d55a6	4	00000718f80fcfd24abb8925c9e6a95
5	0000077403f8b0791e4ed89713e9565	6	000007f30b587fc61bbe123969355a6
7	0000093c5353ce49d36a4f50b516a96	8	000009f4306ad6f086f92cb7d8c96a6
9	00000b34d13a7d09c960d6790ada566	10	00000d3dc8b2c4afaf078dd8678a596
11	00000e6780dc4bb702f1b441fc96965	12	00000eb38c5f53827c9e70716156995

In addition to the canonical $TT(n)$, the maximum number of initial zeroes in our canonical form is also of interest (Note that a zero in the canonical form represents a column of 1s in the Turyn-type sequences). If a method for predicting the number of preceding zeroes could be brought to light, it could greatly decrease future computation for individual Turyn-type sequences, since the first portion of each sequence would be pre-computed. Note that the first entry in each table listed above for $n \leq 32$ represents the maximal number of preceding zeroes for their respective lengths. There seems to be little correlation between n and the maximum number of initial zeros.

By setting $x = 1$ in (1.1), we see that $6n - 2$ is necessarily a sum of six (integer) squares as follows:

$$A(1)^2 + B(1)^2 + 2C(1)^2 + 2D(1)^2 = 6n - 2.$$

It is noteworthy that our computation shows that for all even $n \leq 32$, any choice of four squares $A(1)^2$, $B(1)^2$, $C(1)^2$, $D(1)^2$ satisfying this equation can be realized by some $TT(n)$.

For the sake of completeness, let us mention that $TT(n)$ for $n = 26, 28, 30, 32, 34$ were constructed in [6], and for $n = 36$ in [5]. When transformed to the canonical form (and encoded) these six sequences are:

$$0560110f0f9ec89d54a6867dc$$
$$0005189b4d2e583e5571efc9196$$
$$00788193c52741c99e060a73a22d5$$
$$005088b3dc4d69db0a13438a6c2e916$$
$$052351540cf016cfbe5809958b32825bc$$
$$000f0f51c9bbd750cb048e3902185ca6a96$$

The first four of them indeed occur in our complete listings of class representatives for $n = 26, 28, 30, 32$.

5. The Computational Method

As we have observed, there is compelling computational evidence that $TT(n)$ exist for all even n. While our computational findings positively confirm the existence of $TT(n)$, they also show the difficulty in finding these sequences for large n.

In this section, we describe our method of finding a $TT(38)$ and set the stage for more computational work in trying to find $TT(n)$ for $n \geq 40$.

In order to search for $TT(38)$, we modified the search method in [5]. For the sake of completeness, we will briefly describe our modified search method below.

The search method:

We first find and retain all partial sequences

$$
\begin{aligned}
A^* &= (a_1, \ldots, a_6, a_7, *, \ldots, *, a_{32}, \ldots, a_{38}); \\
B^* &= (b_1, \ldots, b_6, b_7, *, \ldots, *, b_{32}, \ldots, b_{38}); \\
C^* &= (c_1, \ldots, c_6, c_7, *, \ldots, *, c_{32}, \ldots, c_{38}); \\
D^* &= (d_1, \ldots, d_6, *, *, \ldots, *, d_{32}, \ldots, d_{37})
\end{aligned}
$$

for which

$$(N_{A^*} + N_{B^*} + 2N_{C^*} + 2N_{D^*})(s) = 0 \quad \text{for} \quad s \geq 31,$$

and which have the canonical form detailed in Definition 2.1. In order to maintain a feasible number of cases, we precomputed $14, 14, 14, 12$ entries in A, B, C, D respectively. There are 23472940 solutions in total. The set $\mathcal{S}$ of all of these solutions is input for the (modified) algorithm described in [5, p. 438].

To begin: Select a, b, c, d such that

$$a^2 + b^2 + 2c^2 + 2d^2 = 226.$$

Generate all sequences C with the sum of entries equal to c and for which $f_C(\theta) = N_C(0) + 2\sum_{j=1}^{n-1} N_C(j)\cos j\theta \leq 113$ for all $\theta \in \{\frac{j\pi}{600} \mid j = 1, 2, \ldots, 600\}$ and save proper sequences according to their identical first and last seven entries. We do the same for the sequences D with the sum d.

The rest of the procedure is similar to the algorithm in [5]. Choose a solution $\{A^*, B^*, C^*, D^*\}$ in $\mathcal{S}$. Let $\mathcal{C}$ and $\mathcal{D}$ be the sets of those sequences C and D whose first and last seven entries are identical to the first and last seven entries of C^* and D^*, respectively. For any $C \in \mathcal{C}$ and $D \in \mathcal{D}$ for which $f_C(\theta) + f_D(\theta) \leq 113$ for all $\theta \in \{\frac{j\pi}{600} \mid j = 1, 2, \ldots, 600\}$, we proceed to fill in partial sequences A^* and B^* step by step (see [5] for details) until we find appropriate sequences A, B. If we do not find such sequences, then we start again from the beginning.

Our search with $a = 8$, $b = -4$, $c = 8$ and $d = -3$ resulted in the following solution:

$A =$ ++++--+++++-+++---+-++-+++++-++------+
$B =$ +-+++----++-+-++--------+---+++-+-++-+
$C =$ +++-+-+++++-+++-+----+++-+--+--+++-++-
$D =$ +--++---++--++-+----+-+---+-++++-+--+

In encoded form: $05128f55401f041adf7f65c53567822c9cb9c$.

The same algorithm was used to find all representatives of canonical forms. To give a feel of computation time, all $TT(20)$ in canonical form took under five minutes, whereas, all canonical $TT(32)$ took approximately 50000 hours of computations on a single computer.

For the sake of completeness we recall the following known facts. *Base sequences* $(A;B;C;D)$ are quadruples of $\{\pm 1\}$-sequences, with A and B of length m and C and D of length n, and such that

$$N(A)+N(B)+N(C)+N(D)=2(m+n). \tag{5.1}$$

See [1, 2, 3] for details and the classification of these sequences.

Four $\{0,\pm 1\}$-sequences A, B, C, D of length n are called T-*sequences* if

$$(N_A+N_B+N_C+N_D)(s)=0, \quad \text{for} \quad s\geq 1,$$

and in each position, exactly one of the entries of A, B, C, D is nonzero.

If $(A;B;C;D)$ are $TT(n)$, then $(C,D;C,-D;A;B)$ are base sequences of lengths $2n-1$,$2n-1$,n,n, respectively. (Here we use comma as the concatenation operator.) Hence, the existence of $TT(38)$ implies the existence of base sequences of lengths 75, 75, 38, 38.

If $(A;B;C;D)$ are base sequences of lengths m,m,n,n respectively, then

$$((A+B)/2,0_n;(A-B)/2,0_n;0_m,(C+D)/2;0_m,(C-D)/2)$$

are T-sequences of length $m+n$. (Here, the addition and subtraction of two sequences is component-wise, and say 0_m denotes a sequence of m zeroes.) Consequently, T-sequences of length $75+38=113$ exist and we have the following corollary.

Corollary 5.1. *There are base sequences of lengths* 75*,* 75*,* 38*,* 38 *and therefore T-sequences of length* 113*.*

The existence of T-sequences of length 113 implies the existence of an infinite number of Hadamard matrices; see [5] for details.

6. Acknowledgments

D.Ž. Đoković and H. Kharaghani are grateful to NSERC for the continuing support of their research. Part of this work was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET:www.sharcnet.ca), WestGrid and Compute/Calcul Canada. The authors would like to thank an anonymous referee for a detailed list of suggestions which improved the composition and the presentation of the paper considerably.

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