On the Amicability of Orthogonal Designs

W. H. Holzmann and H. Kharaghani*
Department of Mathematics & Computer Science
University of Lethbridge
Lethbridge, Alberta, T1K 3M4
Canada
email: holzmann@uleth.ca, kharaghani@uleth.ca

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Abstract

Although it is known that the maximum number of variables in two amicable orthogonal designs of order $2^n p$, where p is an odd integer, never exceeds $2n+2$, not much is known about the existence of amicable orthogonal designs lacking zero entries that have $2n+2$ variables in total. In this paper we develop two methods to construct amicable orthogonal designs of order $2^n p$ where p odd, with no zero entries and with the total number of variables equal or nearly equal to $2n+2$. In doing so, we make a surprising connection between the two concepts of amicable sets of matrices and an amicable pair of matrices. With the recent discovery of a link between the theory of amicable orthogonal designs and space-time codes, this paper may have applications in space-time codes.

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1 Preliminaries

A *complex orthogonal design* of order n and type $(s_1, s_2, \ldots, s_k)$ denoted $COD(n; s_1, s_2, \ldots, s_k)$ in variables x_1, x_2, …, x_k, is a matrix A of order n with entries in the set

$$\{0, \varepsilon_1 x_1, \varepsilon_2 x_2, \ldots, \varepsilon_k x_k\},$$

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where $\varepsilon_j \in \{\pm 1, \pm i\}$ for each j, which satisfies

$$AA^* = \sum_{j=1}^{k} (s_j x_j^2) I_n,$$

where A^* denotes the conjugate transpose of A and I_n is the identity matrix of order n. A complex orthogonal design in which $\varepsilon_j \in \{\pm 1\}$ for all j is called a (real) orthogonal design and is denoted $OD(n; s_1, s_2, \ldots, s_k)$.

An *amicable pair of complex orthogonal designs*

$$ACOD(n; s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l)$$

of order n and type $(s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l)$ consists of two complex orthogonal designs A and B with

$$A = COD(n; s_1, s_2, \ldots, s_k)$$

and

$$B = COD(n; t_1, t_2, \ldots, t_l)$$

such that $AB^* = BA^*$. In the case of (real) orthogonal designs, that is, when no complex entries are present we use the notation $AOD(n; s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l)$ instead. An amicable pair of orthogonal designs can be used to generate orthogonal designs. We refer the reader to [3, pages 262, 267] and [6, Section 2.] for details.

A well-known method introduced by Goethals and Seidel in 1967 [4], and subsequently extended by Kharaghani [11] has been extensively used to construct orthogonal designs. Let B_j, $j = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_k\}$ satisfying

$$\sum_{j=1}^{4} B_j B_j^t = \sum_{j=1}^{k} (s_j x_j^2) I_n. \tag{1}$$

Then the Goethals-Seidel array

$$GS = \begin{pmatrix} B_1 & B_2R & B_3R & B_4R \\ -B_2R & B_1 & B_4^tR & -B_3^tR \\ -B_3R & -B_4^tR & B_1 & B_2^tR \\ -B_4R & B_3^tR & -B_2^tR & B_1 \end{pmatrix}$$

gives an $OD(4n; s_1, s_2, \ldots, s_k)$, where R is the back-diagonal identity matrix, that is, $R = [r_{jk}]$ where $r_{jk} = 1$ if $j + k = n + 1$ and 0 otherwise. See [3, page 107] for details. Matrices B_j, $j = 1, 2, 3, 4$ satisfying equation (1) are called *type* $(s_1, s_2, \ldots, s_k)$.

It is theoretically impossible to extend the Goethals-Seidel array to an array of order eight without imposing any restrictions on the matrices. One way to restrict variables is as follows. A pair of matrices A, B is said to be *amicable* if $AB^t - BA^t = 0$, and *anti-amicable* if $AB^t + BA^t = 0$. A set $\{A_1, A_2, \ldots, A_{2m}\}$ of square real matrices is said to be *amicable* if

$$\sum_{j=1}^{m} (A_{\sigma(2j-1)} A_{\sigma(2j)}^t - A_{\sigma(2j)} A_{\sigma(2j-1)}^t) = 0$$

for some permutation σ of the set $\{1,2,\ldots,2m\}$. We say that $A_{\sigma(2j-1)}$ *matches* with $A_{\sigma(2j)}$. Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general.

Kharaghani [11] was able to extend the Goethals-Seidel array to an array involving eight variables, which permits the use of an amicable set of eight matrices. A set of matrices $\{A,\tilde{A},B,\tilde{B},C,\tilde{C},D,\tilde{D}\}$ is said to be *special amicable* of type $(s_1, s_2, s_3, s_4; t_1, t_2, t_3, t_4)$ if:

- $\{A,\tilde{A},B,\tilde{B},C,\tilde{C},D,\tilde{D}\}$ is amicable where X matches $\tilde{X}$ for each $X \in \{A,B,C,D\}$,
- A,B,C,D are type (s_1,s_2,s_3,s_4) in variables x_1,x_2,x_3,x_4, and
- $\tilde{A},\tilde{B},\tilde{C},\tilde{D}$ are type (t_1,t_2,t_3,t_4) in variables y_1,y_2,y_3,y_4.

As a nice application of special amicable sets of matrices, it can be shown that each special matching leads to an *infinite* family of orthogonal designs. In Theorem 1 we give a different surprising application.

2 Amicable pairs of orthogonal designs

In this section we introduce a method to generate many classes of amicable pairs of full (no zero entries) orthogonal designs with a maximum number of variables for the first time.

Theorem 1. *If there is a special amicable set of circulant matrices of order n and type* $(s_1,s_2,s_3,s_4;t_1,t_2,t_3,t_4)$ *then there exist:*

$$AOD(8n;\ s_1,s_2,s_3,s_4;\ t_1,t_2,t_3,t_4)$$

and

$$AOD(8n;\ 2s_1,2s_2,2s_3,2s_4;\ 2t_1,2t_2,2t_3,2t_4).$$

Proof. Let $\{A,\tilde{A},B,\tilde{B},C,\tilde{C},D,\tilde{D}\}$ be the special amicable set of matrices where A matches with $\tilde{A}$, B with $\tilde{B}$, etc., $AA^t+BB^t+CC^t+DD^t=(s_1x_1^2+s_2x_2^2+s_3x_3^2+s_4x_4^2)I_n$ and $\tilde{A}\tilde{A}^t+\tilde{B}\tilde{B}^t+\tilde{C}\tilde{C}^t+\tilde{D}\tilde{D}^t=(t_1y_1^2+t_2y_2^2+t_3y_3^2+t_4y_4^2)I_n$. Let $N=I_4\otimes AR$ and $\tilde{N}=I_4\otimes\tilde{A}R$ where R is the order n back-diagonal identity matrix. Set

$$M = \begin{pmatrix} 0 & B & C & D \\ -B & 0 & D^t & -C^t \\ -C & -D^t & 0 & B^t \\ -D & C^t & -B^t & 0 \end{pmatrix},$$

$$\tilde{M} = \begin{pmatrix} 0 & \tilde{B} & \tilde{C} & \tilde{D} \\ -\tilde{B} & 0 & -\tilde{D}^t & \tilde{C}^t \\ -\tilde{C} & \tilde{D}^t & 0 & -\tilde{B}^t \\ -\tilde{D} & -\tilde{C}^t & \tilde{B}^t & 0 \end{pmatrix}.$$

Consider the 2 by 2 matrices,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ - & 0 \end{pmatrix}, \quad \text{and } Q = PS = -SP = \begin{pmatrix} 1 & 0 \\ 0 & - \end{pmatrix}.$$

Let

$$\begin{aligned} U &= N \otimes I + M \otimes S, \\ \tilde{U} &= \tilde{N} \otimes Q + \tilde{M} \otimes P. \end{aligned}$$

Then $U, \tilde{U}$ form a pair of amicable orthogonal designs

$$AOD(8n;\ s_1, s_2, s_3, s_4;\ t_1, t_2, t_3, t_4),$$

as we now show. The fact that U is an $OD(8n;\ s_1, s_2, s_3, s_4)$ follows since $UU^t = (NN^t + MM^t) \otimes I + (NM^t + MN^t) \otimes S$; here $NN^t + MM^t = I_4 \otimes (AA^t + BB^t + CC^t + DD^t) = I_4 \otimes (s_1x_1^2 + s_2x_2^2 + s_3x_3^2 + s_4x_4^2)I_n$ by assumption, while $NM^t + MN^t = 0$ since NM^t is skew (which can be established by using the fact that products such as ARB are symmetric). A similar argument shows that $\tilde{U}$ is an $OD(8n;\ t_1,\ t_2,\ t_3,\ t_4)$. To show that $U\tilde{U}^t = \tilde{U}U^t$ we expand the product $U\tilde{U}^t$ to see how the fact that the matrices form an amicable set of matrices is used. First note that

$$U\tilde{U}^t = (N\tilde{N}^t + M\tilde{M}^t) \otimes Q - (M\tilde{N}^t + N\tilde{M}^t) \otimes P.$$

We need to show that $U\tilde{U}^t$ is a symmetric matrix. Since Q is a symmetric matrix and P is skew, we need to show that the matrix $N\tilde{N}^t + M\tilde{M}^t =$

$$\begin{pmatrix} f(A,B) + f(C,D) & -C\tilde{D} + D\tilde{C} & B\tilde{D} - D\tilde{B} & -B\tilde{C} + C\tilde{B} \\ D^t\tilde{C}^t - C^t\tilde{D}^t & f(A,B) - g(C,D) & B\tilde{C}^t + C^t\tilde{B} & B\tilde{D}^t + D^t\tilde{B} \\ -D^t\tilde{B}^t + B^t\tilde{D}^t & C\tilde{B}^t + B^t\tilde{C} & f(A,C) - g(B,D) & C\tilde{D}^t + D^t\tilde{C} \\ C^t\tilde{B}^t - B^t\tilde{C}^t & D\tilde{B}^t + B^t\tilde{D} & D\tilde{C}^t + C^t\tilde{D} & f(A,D) - g(B,C) \end{pmatrix}$$

where $f(X,Y) = X\tilde{X}^t + Y\tilde{Y}^t$ and $g(X,Y) = X^t\tilde{X} + Y^t\tilde{Y}$ is symmetric and the matrix

$$M\tilde{N}^t + N\tilde{M}^t = \begin{pmatrix} 0 & BR\tilde{A}^t - ARB\tilde{B}^t & CR\tilde{A}^t - AR\tilde{C}^t & DR\tilde{A}^t - AR\tilde{D}^t \\ -BR\tilde{A}^t + AR\tilde{B}^t & 0 & D^tR\tilde{A}^t + AR\tilde{D} & -C^tR\tilde{A}^t - AR\tilde{C} \\ -CR\tilde{A}^t + AR\tilde{C}^t & -D^tR\tilde{A}^t - AR\tilde{D} & 0 & B^tR\tilde{A}^t + AR\tilde{B} \\ -DR\tilde{A}^t + AR\tilde{D}^t & C^tR\tilde{A}^t + AR\tilde{C} & -B^tR\tilde{A}^t - AR\tilde{B} & 0 \end{pmatrix}$$

is skew. Noting that all the matrices $A, B, C, D, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are circulant, we must only show that each of the diagonal blocks of $N\tilde{N}^t + M\tilde{M}^t$ are symmetric, and this reduces to the single requirement that:

$$A\tilde{A}^t - \tilde{A}A^t + B\tilde{B}^t - \tilde{B}B^t + C\tilde{C}^t - \tilde{C}C^t + D\tilde{D}^t - \tilde{D}D^t = 0.$$

But this is exactly the requirement that the matrices $\{A, \tilde{A}, B, \tilde{B}, C, \tilde{C}, D, \tilde{D}\}$ form an amicable set of matrices where A matches with $\tilde{A}$, B with $\tilde{B}$, etc. This completes the proof of the first part.

For the second part of the theorem we take:

$$\begin{aligned} U &= N \otimes H + M \otimes SH, \\ \tilde{U} &= \tilde{N} \otimes QH + \tilde{M} \otimes PH, \end{aligned}$$

where $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The rest of the proof is similar to the proof of the first part of the theorem. □

Remark 1.

- We have found many special amicable sets of circulant matrices in [6, 7, 8, 9, 10]. Each of these sets can be used to generate new pairs of amicable orthogonal designs.
- One can use $I_m \otimes I$, $I_m \otimes P$, $I_m \otimes Q$, $I_m \otimes R$ and a complex Hadamard matrix C_{2m} of order $2m$ instead of I, P, Q, R and H respectively to get many amicable pairs of complex orthogonal designs.
- In a frequently referenced paper [14], Tarokh, Jafarkhani and Calderbank show a link between the theory of amicable orthogonal designs and space-time codes. The results of this paper may have applications in space-time codes.
- There is a connection between the concept of *product designs* introduced by Robinson [3, 13] and the construction here; this will be discussed in a forthcoming paper.

Next we present an example that generates new pairs of amicable orthogonal designs of order 24. This example shows the power of the method presented here. As usual for brevity, by $(a_1, a_2, \dots, a_k)$ we mean a circulant matrix with the first row $a_1, a_2, \dots, a_k$.

Example 1.

Consider the special amicable set $\{A = (a, b, c), B = (-b, a, d), C = (-c, -d, a), D = (d, -c, b), \tilde{A} = (e, f, g), \tilde{B} = (-g, -h, e), \tilde{C} = (-f, e, h), \tilde{D} = (-h, g, -f)\}$, which in fact gives rise to an $OD(24; 3, 3, 3, 3, 3, 3, 3, 3)$, an instance of a Plotkin array [8, 9, 12]. By Theorem 1 we obtain the following new AODs:

$$AOD(24;\ 3, 3, 3, 3;\ 3, 3, 3, 3),$$
$$AOD(24;\ 6, 6, 6, 6;\ 6, 6, 6, 6).$$

These designs are shown in Tables 1 and 2.

A theorem of Wolfe [3, p. 227 Cor 5.32] says that the maximum total number of variables in the two matrices of an AOD of order $2^\alpha p$, p odd, is $2\alpha + 2$. For the case of $24 = 2^3 \cdot 3$ the maximum number of variables is $2 \cdot 3 + 2 = 8$. These examples achieve the bound. Furthermore, there are no zero entries in the second AOD, which is obtained from the first AOD by the construction given in Theorem 1.

Table 1: An AOD(24; 3,3,3,3; 3,3,3,3) where $A = -a$, $B = -b$, etc.

c	0	b	0	a	0	0	B	0	a	0	d	0	C	0	D	0	a	0	d	0	C	0	b
0	c	0	b	0	a	B	0	a	0	d	0	C	0	D	0	a	0	d	0	C	0	b	0
b	0	a	0	c	0	0	d	0	B	0	a	0	a	0	C	0	D	0	b	0	d	0	C
0	b	0	a	0	c	d	0	B	0	a	0	a	0	C	0	D	0	b	0	d	0	C	0
a	0	c	0	b	0	0	a	0	d	0	B	0	D	0	a	0	C	0	C	0	b	0	d
0	a	0	c	0	b	a	0	d	0	B	0	D	0	a	0	C	0	C	0	b	0	d	0
0	b	0	A	0	D	c	0	b	0	a	0	0	d	0	b	0	C	0	c	0	A	0	d
b	0	A	0	D	0	0	c	0	b	0	a	d	0	b	0	C	0	c	0	A	0	d	0
0	D	0	b	0	A	b	0	a	0	c	0	0	C	0	d	0	b	0	d	0	c	0	A
D	0	b	0	A	0	0	b	0	a	0	c	C	0	d	0	b	0	d	0	c	0	A	0
0	A	0	D	0	b	a	0	c	0	b	0	0	b	0	C	0	d	0	A	0	d	0	c
A	0	D	0	b	0	0	a	0	c	0	b	b	0	C	0	d	0	A	0	d	0	c	0
0	c	0	d	0	A	0	D	0	B	0	c	c	0	b	0	a	0	0	B	0	d	0	a
c	0	d	0	A	0	D	0	B	0	c	0	0	c	0	b	0	a	B	0	d	0	a	0
0	A	0	c	0	d	0	c	0	D	0	B	b	0	a	0	c	0	0	a	0	B	0	d
A	0	c	0	d	0	c	0	D	0	B	0	0	b	0	a	0	c	a	0	B	0	d	0
0	d	0	A	0	c	0	B	0	c	0	D	a	0	c	0	b	0	0	d	0	a	0	B
d	0	A	0	c	0	B	0	c	0	D	0	0	a	0	c	0	b	d	0	a	0	B	0
0	D	0	c	0	B	0	C	0	a	0	D	0	b	0	D	0	A	c	0	b	0	a	0
D	0	c	0	B	0	C	0	a	0	D	0	b	0	D	0	A	0	0	c	0	b	0	a
0	B	0	D	0	c	0	D	0	C	0	a	0	A	0	b	0	D	b	0	a	0	c	0
B	0	D	0	c	0	D	0	C	0	a	0	A	0	b	0	D	0	0	b	0	a	0	c
0	c	0	B	0	D	0	a	0	D	0	C	0	D	0	A	0	b	a	0	c	0	b	0
c	0	B	0	D	0	a	0	D	0	C	0	D	0	A	0	b	0	0	a	0	c	0	b

g	0	f	0	e	0	0	G	0	H	0	e	0	F	0	e	0	h	0	H	0	g	0	F
0	G	0	F	0	E	g	0	h	0	E	0	f	0	E	0	H	0	h	0	G	0	f	0
f	0	e	0	g	0	0	e	0	G	0	H	0	h	0	F	0	e	0	F	0	H	0	g
0	F	0	E	0	G	E	0	g	0	h	0	H	0	f	0	E	0	f	0	h	0	G	0
e	0	g	0	f	0	0	H	0	e	0	G	0	e	0	h	0	F	0	g	0	F	0	H
0	E	0	G	0	F	h	0	E	0	g	0	E	0	H	0	f	0	G	0	f	0	h	0
0	g	0	h	0	E	g	0	f	0	e	0	0	h	0	f	0	G	0	F	0	h	0	e
G	0	H	0	e	0	0	G	0	F	0	E	H	0	F	0	g	0	f	0	H	0	E	0
0	E	0	g	0	h	f	0	e	0	g	0	0	G	0	h	0	f	0	e	0	F	0	h
e	0	G	0	H	0	0	F	0	E	0	G	g	0	H	0	F	0	E	0	f	0	H	0
0	h	0	E	0	g	e	0	g	0	f	0	0	f	0	G	0	h	0	h	0	e	0	F
H	0	e	0	G	0	0	E	0	G	0	F	F	0	g	0	H	0	H	0	E	0	f	0
0	f	0	E	0	H	0	H	0	F	0	g	g	0	f	0	e	0	0	g	0	E	0	h
F	0	e	0	h	0	h	0	f	0	G	0	0	G	0	F	0	E	G	0	e	0	H	0
0	H	0	f	0	E	0	g	0	H	0	F	f	0	e	0	g	0	0	h	0	g	0	E
h	0	F	0	e	0	G	0	h	0	f	0	0	F	0	E	0	G	H	0	G	0	e	0
0	E	0	H	0	f	0	F	0	g	0	H	e	0	g	0	f	0	0	E	0	h	0	g
e	0	h	0	F	0	f	0	G	0	h	0	0	E	0	G	0	F	e	0	H	0	G	0
0	h	0	G	0	f	0	f	0	H	0	E	0	G	0	e	0	H	g	0	f	0	e	0
H	0	g	0	F	0	F	0	h	0	e	0	g	0	E	0	h	0	0	G	0	F	0	E
0	f	0	h	0	G	0	E	0	f	0	H	0	H	0	G	0	e	f	0	e	0	g	0
F	0	H	0	g	0	e	0	F	0	h	0	h	0	g	0	E	0	0	F	0	E	0	G
0	G	0	f	0	h	0	H	0	E	0	f	0	e	0	H	0	G	e	0	g	0	f	0
g	0	F	0	H	0	h	0	e	0	F	0	E	0	h	0	g	0	0	E	0	G	0	F

Table 2: An AOD(24; 6,6,6,6; 6,6,6,6) where $A = -a$, $B = -b$, etc.

c	c	b	b	a	a	B	b	a	A	d	D	C	c	D	d	a	A	d	D	C	c	b	B
c	C	b	B	a	A	B	B	a	a	d	d	C	C	D	D	a	a	d	d	C	C	b	b
b	b	a	a	c	c	d	D	B	b	a	A	a	A	C	c	D	d	b	B	d	D	C	c
b	B	a	A	c	C	d	d	B	B	a	a	a	a	C	C	D	D	b	b	d	d	C	C
a	a	c	c	b	b	a	A	d	D	B	b	D	d	a	A	C	c	C	c	b	B	d	D
a	A	c	C	b	B	a	a	d	d	B	B	D	D	a	a	C	C	C	C	b	b	d	d
b	B	A	a	D	d	c	c	b	b	a	a	d	D	b	B	C	c	c	C	A	a	d	D
b	b	A	A	D	D	c	C	b	B	a	A	d	d	b	b	C	C	c	c	A	A	d	d
D	d	b	B	A	a	b	b	a	a	c	c	C	c	d	D	b	B	d	D	c	C	A	a
D	D	b	b	A	A	b	B	a	A	c	C	C	C	d	d	b	b	d	d	c	c	A	A
A	a	D	d	b	B	a	a	c	c	b	b	b	B	C	c	d	D	A	a	d	D	c	C
A	A	D	D	b	b	a	A	c	C	b	B	b	b	C	C	d	d	A	A	d	d	c	c
c	C	d	D	A	a	D	d	B	b	c	C	c	c	b	b	a	a	B	b	d	D	a	A
c	c	d	d	A	A	D	D	B	B	c	c	c	C	b	B	a	A	B	B	d	d	a	a
A	a	c	C	d	D	c	C	D	d	B	b	b	b	a	a	c	c	a	A	B	b	d	D
A	A	c	c	d	d	c	c	D	D	B	B	b	B	a	A	c	C	a	a	B	B	d	d
d	D	A	a	c	C	B	b	c	C	D	d	a	a	c	c	b	b	d	D	a	A	B	b
d	d	A	A	c	c	B	B	c	c	D	D	a	A	c	C	b	B	d	d	a	a	B	B
D	d	c	C	B	b	C	c	a	A	D	d	b	B	D	d	A	a	c	c	b	b	a	a
D	D	c	c	B	B	C	C	a	a	D	D	b	b	D	D	A	A	c	C	b	B	a	A
B	b	D	d	c	C	D	d	C	c	a	A	A	a	b	B	D	d	b	b	a	a	c	c
B	B	D	D	c	c	D	D	C	C	a	a	A	A	b	b	D	D	b	B	a	A	c	C
c	C	B	b	D	d	a	A	D	d	C	c	D	d	A	a	b	B	a	a	c	c	b	b
c	c	B	B	D	D	a	a	D	D	C	C	D	D	A	A	b	b	a	A	c	C	b	B

g	g	f	f	e	e	G	g	H	h	e	E	F	f	e	E	h	H	H	h	g	G	F	f
G	g	F	f	E	e	g	g	h	h	E	E	f	f	E	E	H	H	h	h	G	G	f	f
f	f	e	e	g	g	e	E	G	g	H	h	h	H	F	f	e	E	F	f	H	h	g	G
F	f	E	e	G	g	E	E	g	g	h	h	H	H	f	f	E	E	f	f	h	h	G	G
e	e	g	g	f	f	H	h	e	E	G	g	e	E	h	H	F	f	g	G	F	f	H	h
E	e	G	g	F	f	h	h	E	E	g	g	E	E	H	H	f	f	G	G	f	f	h	h
g	G	h	H	E	e	g	g	f	f	e	e	h	H	f	F	G	g	F	f	h	H	e	E
G	G	H	H	e	e	G	g	F	f	E	e	H	H	F	F	g	g	f	f	H	H	E	E
E	e	g	G	h	H	f	f	e	e	g	g	G	g	h	H	f	F	e	E	F	f	h	H
e	e	G	G	H	H	F	f	E	e	G	g	g	g	H	H	F	F	E	E	f	f	H	H
h	H	E	e	g	G	e	e	g	g	f	f	f	F	G	g	h	H	h	H	e	E	F	f
H	H	e	e	G	G	E	e	G	g	F	f	F	F	g	g	H	H	H	H	E	E	f	f
f	F	E	e	H	h	H	h	F	f	g	G	g	g	f	f	e	e	g	G	E	e	h	H
F	F	e	e	h	h	h	h	f	f	G	G	G	g	F	f	E	e	G	G	e	e	H	H
H	h	f	F	E	e	g	G	H	h	F	f	f	f	e	e	g	g	h	H	g	G	E	e
h	h	F	F	e	e	G	G	h	h	f	f	F	f	E	e	G	g	H	H	G	G	e	e
E	e	H	h	f	F	F	f	g	G	H	h	e	e	g	g	f	f	E	e	h	H	g	G
e	e	h	h	F	F	f	f	G	G	h	h	E	e	G	g	F	f	e	e	H	H	G	G
h	H	G	g	f	F	f	F	H	h	E	e	G	g	e	E	H	h	g	g	f	f	e	e
H	H	g	g	F	F	F	F	h	h	e	e	g	g	E	E	h	h	G	g	F	f	E	e
f	F	h	H	G	g	E	e	f	F	H	h	H	h	G	g	e	E	f	f	e	e	g	g
F	F	H	H	g	g	e	e	F	F	h	h	h	h	g	g	E	E	F	f	E	e	G	g
G	g	f	F	h	H	H	h	E	e	f	F	e	E	H	h	G	g	e	e	g	g	f	f
g	g	F	F	H	H	h	h	e	e	F	F	E	E	h	h	g	g	E	e	G	g	F	f

3 Application: Some new orthogonal designs

An amicable pair of orthogonal designs of order n can be used to construct an orthogonal design of order $4n$. Applying the method in this paper, we first searched for all full amicable orthogonal designs of order 24 involving eight variables obtaining: $AOD(24;\ 2,2,4,16;\ 2,2,4,16)$, $AOD(24;\ 2,2,10,10;\ 2,2,10,10)$, $AOD(24;\ 2,4,6,12;\ 2,4,6,12)$, $AOD(24;\ 4,4,8,8;\ 4,4,8,8)$, and $AOD(24;\ 6,6,6,6;\ 6,6,6,6)$. Using these and known construction methods we generated orthogonal designs of order 96 involving 10 variables. Note that by theory 10 is the maximum possible number of variables in an OD of order 96. In total we were able to generate 31 new full ODs of order 96 in 10 variables. Their types are listed in Table 3. We use the standard notation, a_ℓ, instead of repeating a ℓ-times.

Table 3: Types of new full ODs of order 96 in 10 variables

$(6_7,18_3)$, $(4_2,8_5,12_2,24)$, $(4_5,8_2,20_3)$, $(4_5,8_2,12,24_2)$, $(2,4,6,12_7)$, $(2,4,6,10_3,12,14_3)$, $(2,4,6,8_3,12,16_3)$, $(2,4,6_2,12_5,18)$, $(2,4,6_4,12,18_3)$, $(2,4,6_5,12_2,36)$, $(2,4_2,8_4,10_3)$, $(2,4_2,6,8_4,12_2)$, $(2,4_4,6,12,20_3)$, $(2,4_4,6_2,12,18,36)$, $(2,4_5,8,14_3)$, $(2,4_5,6,8,12,24)$, $(2_2,10_2,12_6)$, $(2_2,10_5,14_3)$, $(2_2,6_2,10_5,30)$, $(2_2,4,6_2,12,16_4)$, $(2_2,4,6_3,16,18_3)$, $(2_2,4_4,16,20_3)$, $(2_2,4_4,6_2,16,48)$, $(2_4,4,6,12,22_3)$, $(2_4,4,6,12_2,18,36)$, $(2_4,4_2,8,16_3)$, $(2_4,4_2,8,12_2,24)$, $(2_5,10_2,22_3)$, $(2_5,6,10_2,30_2)$, $(2_5,4,16,22_3)$, $(2_5,4,6,12,16,48)$

4 The existence of amicable orthogonal arrays

Not much is known about the existence or the structure of full orthogonal designs with a maximum number of variables. In order to investigate the existence of orthogonal designs, there is a need for plug-in arrays similar to Goethals-Seidel arrays. In this section we find some new arrays; see Theorem 2.

We begin with a simple example of a pair of orthogonal designs of small order with the maximum possible number of variables.

Example 2.

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \qquad B = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}$$

is an $AOD(2;1,1;1,1)$.

As noted in [3], this remarkable pair of matrices is sufficient to provide many sets of matrices with very useful properties as follows:

Lemma 1 (Wolfe [15])**.** *Given an integer $n = 2^s u$ where u is odd, $s \geq 1$, there are two sets $\mathcal{P} = \{P_0, \ldots, P_s\}$ and $\mathcal{Q} = \{Q_0, \ldots, Q_s\}$ of signed permutation matrices of order n such that:*

i. $\mathcal{P}$ consists of disjoint pairwise anti-amicable matrices,

ii. $\mathcal{Q}$ consists of disjoint pairwise anti-amicable matrices, and

iii. for each i and j, $P_i Q_j^t = Q_j P_i^t$.

Proof. Compare with [5, Lemma 6] in which another related set is given (but note that Q_{11} and Q_{12} should be interchanged there).

Let

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ - & 0 \end{pmatrix}, \quad \text{and } Q = PS = -SP = \begin{pmatrix} 1 & 0 \\ 0 & - \end{pmatrix}.$$

These matrices can be obtained from Example 2 by writing $A = aI + bP$ and $B = xQ + yS$. Let R be the back-diagonal identity matrix of order u. The required sets $\mathcal{P}$ and $\mathcal{Q}$ are found as follows. Define $P_0 = (\otimes_{i=1}^{s} I) \otimes R$, $P_k = (\otimes_{i=1}^{k-1} I) \otimes P \otimes (\otimes_{i=k+1}^{s} S) \otimes R$ for $0 < k \leq s$, $Q_0 = (\otimes_{i=1}^{s} S) \otimes R$, and $Q_k = (\otimes_{i=1}^{k-1} I) \otimes Q \otimes (\otimes_{i=k+1}^{s} S) \otimes R$ for $0 < k \leq s$.

The pairwise anti-amicable property of part 1 of this lemma follows because I and P are pairwise anti-amicable, as are $P \otimes S$ and $I \otimes P$. Similarly for part 2 of this lemma but use the fact that S and Q are pairwise anti-amicable as are $Q \otimes S$ and $I \otimes Q$. The case $0 < i < j$ in part 3 of this lemma follows because $P \otimes S$ and $I \otimes Q$ are amicable, the case $0 < j < i$ follows because $Q \otimes S$ and $I \otimes P$ are amicable, and the case $0 < i = j$ follows because P and Q are amicable. The remaining cases can be easily handled. □

A set of *near type 1* matrices is a set $\mathcal{C}$ of commuting matrices with $bc^t = c^t b$ for all $b, c \in \mathcal{C}$. Near type 1 matrices include order one matrices in a single variable, order two circulant or negacirculant matrices in two variables, and the order four matrices presented in Remark 2 below.

We now embark on a method, similar in spirit to the method first developed in [2]. Given the odd integer n we write the binary expansion of n as $n = 1 + \varepsilon 2 + 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_m}$, where $\varepsilon \in \{0, 1\}$, and $1 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_m$. Before giving the general result, by way of illustration we consider the case $n = 9 = 1 + 2^3$. So $\varepsilon = 0$, $\alpha_1 = 3$ and $m = 1$.

Example 3.

Let $(x_1, \ldots, x_n)$ denote the circulant matrix with first row $x_1, \ldots, x_n$. Consider the following circulant matrices A_0, A_1 and A_2:

$$\begin{aligned} A_0 &= (a, 0, 0, 0, 0, 0, 0, 0, 0), \\ A_1 &= (0, b, c, 0, 0, 0, 0, 0, 0), \\ A_2 &= (0, 0, 0, b, -c, 0, 0, 0, 0), \end{aligned}$$

where a, b and c are near type 1 matrices, and 0 is the zero matrix of the same order. Construct symmetric matrices S_j and Hermitian matrices H_j as follows:

$$\begin{aligned} S_0 = {} & \tfrac{1}{2}(A_0 + A_0^t) && = \tfrac{1}{2}(a + a^t, 0, 0, 0, 0, 0, 0, 0, 0), \\ H_0 = {} & \tfrac{i}{2}(A_0 - A_0^t) && = \tfrac{i}{2}(a - a^t, 0, 0, 0, 0, 0, 0, 0, 0), \\ S_1 = {} & A_1 + A_1^t && = (0, b, c, 0, 0, 0, 0, c^t, b^t), \\ H_1 = {} & i(A_1 - A_1^t) && = (0, ib, ic, 0, 0, 0, 0, -ic^t, -ib^t), \\ S_2 = {} & A_2 + A_2^t && = (0, 0, 0, b, -c, -c^t, b^t, 0, 0), \\ H_2 = {} & i(A_2 - A_2^t) && = (0, 0, 0, ib, -ic, ic^t, -ib^t, 0, 0). \end{aligned}$$

Replace a, b, and c with $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ respectively, giving matrices $\tilde{A}_0$, $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{S}_0$, $\tilde{S}_1$, $\tilde{S}_2$, $\tilde{H}_0$, $\tilde{H}_1$, and $\tilde{H}_2$. Let H be a Hadamard matrix of order 2^5, such as

$$H = \otimes_{j=1}^{5} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Construct the matrices $\{P_0,\ldots,P_5\}$ and $\{Q_0,\ldots,Q_5\}$ of Lemma 1 for the integer 2^5.
Let

$$\begin{aligned} A &= S_0\otimes P_0H + H_0\otimes P_1H \\ &\quad + S_1\otimes\frac{1}{2}(P_2+P_3)H + H_1\otimes\frac{1}{2}(P_2-P_3)H \\ &\quad + S_2\otimes\frac{1}{2}(P_4+P_5)H + H_2\otimes\frac{1}{2}(P_4-P_5)H, \end{aligned}$$

and similarly

$$\begin{aligned} B &= \tilde{S}_0\otimes Q_0H + \tilde{H}_0\otimes Q_1H \\ &\quad + \tilde{S}_1\otimes\frac{1}{2}(Q_2+Q_3)H + \tilde{H}_1\otimes\frac{1}{2}(Q_2-Q_3)H \\ &\quad + \tilde{S}_2\otimes\frac{1}{2}(Q_4+Q_5)H + \tilde{H}_2\otimes\frac{1}{2}(Q_4-Q_5)H. \end{aligned}$$

Then, as we will show in the proof of Theorem 2, both A and B are complex orthogonal arrays each involving three near type 1 matrices such that $AB^*=BA^*$. If the near type 1 matrices are of order one, then we obtain an $ACOD(288;32,128_2;\,32,128_2)$ in six variables a, b, c, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$. If the near type 1 matrices are negacyclic of order two in different variables, that is, of the form:

$$\begin{pmatrix} x_j & y_j \\ -y_j & x_j \end{pmatrix},$$

then we obtain an $ACOD(576;\,32_2,128_4;\,32_2,128_4)$ in 12 variables.

A pair of two variable *Golay sequences* of length 2^j, $j\geq 0$, can be constructed inductively as follows. $(b;c)$ is the sequence of length 1. The sequence of length 2^j, $j>0$ is $(X,Y;X,-Y)$ where $(X;Y)$ is the sequence of length 2^{j-1} and "," denotes concatenation of sequences; see also [2]. In Example 3, the Golay sequence $(b,c;b,-c)$ was used to define A_1 and A_2.

Theorem 2. *For every odd integer n where $n=1+\varepsilon 2+2^{\alpha_1}+2^{\alpha_2}+\cdots+2^{\alpha_m}$, with $\varepsilon\in\{0,1\}$, and $1<\alpha_1<\alpha_2<\cdots<\alpha_m$, there is a pair A, B of arrays each involving $s=1+\varepsilon+2m$ near type 1 matrices such that:*

$$\begin{aligned} AA^* &= 2^{2s-1}\left(x_0^2+2\varepsilon x_{2m+1}^2+\sum_{j=1}^{m}2^{\alpha_j-1}(x_{2j-1}^2+x_{2j}^2)\right)I_{2^{2s-1}n}, \\ BB^* &= 2^{2s-1}\left(\tilde{x}_0^2+2\varepsilon\tilde{x}_{2m+1}^2+\sum_{j=1}^{m}2^{\alpha_j-1}(\tilde{x}_{2j-1}^2+\tilde{x}_{2j}^2)\right)I_{2^{2s-1}n}, \\ AB^* &= BA^*, \end{aligned}$$

where x_0, …, x_{2m}, (and also x_{2m+1} if $\varepsilon=1$) are the s matrices in A and $\tilde{x}_0$, …, $\tilde{x}_{2m}$, (and also $\tilde{x}_{2m+1}$ if $\varepsilon=1$) are the s matrices in B. This pair of arrays can be used to give an amicable pair of complex orthogonal designs of order $2^{2s-1}n$ each involving s variables, and an amicable pair of complex orthogonal designs of order $2^{2s}n$ each involving $2s$ variables.

Proof. We generalize the construction of Example 3.

Let $\{x_0,\dots,x_{s-1}\}$ be a collection of near type 1 matrices. Define $A_0=(x_0,0_{n-1})$. Let $(X;Y)$ be a pair of Golay sequences of length 2^{α_j-2} in x_{2j-1}, x_{2j} for $j=1,\dots,m$ and construct

$$\begin{aligned} A_{2j-1} &= (0_{1+\varepsilon+2^{\alpha_1-1}+\cdots+2^{\alpha_{j-1}-1}}, X, 0_{2^{\alpha_j-2}}, 0_{2^{\alpha_{j+1}-1}+\cdots+2^{\alpha_m-1}+\frac{1}{2}(n-1)}),\\ A_{2j} &= (0_{1+\varepsilon+2^{\alpha_1-1}+\cdots+2^{\alpha_{j-1}-1}}, 0_{2^{\alpha_j-2}}, Y, 0_{2^{\alpha_{j+1}-1}+\cdots+2^{\alpha_m-1}+\frac{1}{2}(n-1)}). \end{aligned}$$

If $\varepsilon=1$ also take $A_{2m+1}=(0,x_{2m+1},0_{n-2})$. Thus we have s matrices A_k in s variables x_k, $k=0,\dots,s-1$.

Use a Hadamard matrix H of order 2^{2s-1} and the matrices of $\mathcal{P}=\{P_0,P_1,\dots,P_{2s-1}\}$, $\mathcal{Q}=\{Q_0,Q_1,\dots,Q_{2s-1}\}$ of Lemma 1.

Construct:

$$\begin{aligned} S_0 &= \frac{1}{2}(A_0+A_0^t),\\ H_0 &= \frac{i}{2}(A_0-A_0^t), \end{aligned}$$

and for $k=1,\dots,s-1$:

$$\begin{aligned} S_k &= A_k+A_k^t,\\ H_k &= i(A_k-A_k^t). \end{aligned}$$

Define

$$\begin{aligned} A &= S_0\otimes P_0H+H_0\otimes P_1H && (2)\\ &\quad+\sum_{k=1}^{s-1}\left(S_k\otimes\frac{1}{2}(P_{2k}+P_{2k+1})H+H_k\otimes\frac{1}{2}(P_{2k}-P_{2k+1})H\right). && (3) \end{aligned}$$

To show that AA^* is a multiple of an identity matrix, use the fact that $\{P_0,P_1\}\cup\{P_{2k}\pm P_{2k+1}:1\le k<s\}$ is a collection of anti-amicable matrices and the set of all S_k and H_k form a commuting set of Hermitian matrices; also because Golay sequences are involved, simplify $A_{2j-1}A_{2j-1}^t+A_{2j}A_{2j}^t=2^{\alpha_j-2}(x_{2j-1}^2+x_{2j}^2)I_n$ to obtain the expression for AA^* given above.

Replace each x_k with $\tilde{x}_k$ giving matrices $\tilde{A}_k$, $\tilde{S}_k$, and $\tilde{H}_k$ and define

$$\begin{aligned} B &= \tilde{S}_0\otimes Q_0H+\tilde{H}_0\otimes Q_1H\\ &\quad+\sum_{k=1}^{s-1}\left(\tilde{S}_k\otimes\frac{1}{2}(Q_{2k}+Q_{2k+1})H+\tilde{H}_k\otimes\frac{1}{2}(Q_{2k}-Q_{2k+1})H\right). \end{aligned}$$

Then BB^* is a multiple of an identity matrix and $AB^*=BA^*$ by arguments similar to those given above for AA^*.

Note that the entries of A are in $\mathcal{X} = \{0, \pm x_k, \pm ix_k : 0 \le k < s\}$ as long as the entries for S_0 and H_0 are in $\mathcal{X}$. Clearly the entries of summands of (2) are in $\mathcal{X}$. For the summands of (3), S_k, H_k, $k > 0$, have entries in $\mathcal{X}$ since A_k and A_k^t are disjoint, while $\frac{1}{2}(P_{2k} \pm P_{2k+1})H$ have entries in $\{0, \pm 1\}$ since $\mathcal{P}$ is a collection of disjoint signed permutation matrices. If the near type 1 matrices are negacirculant matrices are of order one, then we obtain an amicable pair of complex orthogonal designs of order $2^{2s-1}n$ each involving s variables. If the near type 1 matrices are negacirculant matrices of order two in different variables, that is, of the form:

$$\begin{pmatrix} x_k & y_k \\ -y_k & x_k \end{pmatrix},$$

then we obtain an amicable pair of complex orthogonal designs of order $2^{2s}n$ each involving $2s$ variables. □

Remark 2.

- The CODs obtained in the previous theorem have *almost* the maximal possible number of free variables.
- It is possible to make use of a pair of complex Golay sequences (see [1] for details) instead of a pair of Golay sequences in our construction. For the sake of brevity we omit this.
- Matrices of form

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_2 & x_2 \\ -x_2 & x_1 & x_2 & -x_2 \\ -x_2 & -x_2 & x_1 & x_2 \\ -x_2 & x_2 & -x_2 & x_1 \end{pmatrix}$$

  are near type 1 matrices. Example 3 extends to give an $ACOD(1152; 32, 96, 128_2, 384_2; 32, 96, 128_2, 384_2)$ in 12 variables.

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