Unbiased complex Hadamard matrices and bases

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Abstract

We introduce mutually unbiased complex Hadamard (**MUCH**) matrices and show that the number of **MUCH** matrices of order $2n$, n odd, is at most 2 and the bound is attained for $n = 1, 5, 9$. Furthermore, we prove that certain pairs of mutually unbiased complex Hadamard matrices of order m can be used to construct pairs of unbiased real Hadamard matrices of order $2m$. As a consequence we generate a new pair of unbiased real Hadamard matrices of order 36.

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1 Preliminaries

A *complex Hadamard* matrix is a matrix H of order n with entries in $\{-1, 1, i, -i\}$ and orthogonal rows in the usual complex inner product on $\mathbb{C}^n$. If the entries of the matrix consist of only ± 1, we call the matrix a real Hadamard matrix or a Hadamard matrix for short. Our main references for complex and real Hadamard matrices are [7, 8]. Two complex Hadamard matrices H and K of order $2n$ are called *unbiased* if $HK^* = L$, where K^* denotes the Hermitian transpose of K and all the entries of the matrix L are of the absolute value $\sqrt{2n}$. In this case, it follows that $2n = a^2 + b^2$, where a, b are nonnegative integers. While there has been a lot of interest

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in the class of mutually unbiased unimodular complex Hadamard matrices, where the entries of the matrices consist of unimodular complex numbers, see [2, 3, 9] for details, it is only recently that some interest has been shown in the existence and applications of mutually unbiased real Hadamard matrices, see [5]. Our aim in this paper is to concentrate on matrices of order $2n$, n odd, with entries in $\{-1,1,i,-i\}$. We will find an upper bound for the number of mutually unbiased complex Hadamard matrices of order $2n$, n odd, denoted $|\mathbf{MUCH}(2n)|$, in the next section. We also report on the outcome of a computer search for maximal classes of **MUCH** matrices of orders 10 and 18. Section 3 is devoted to the study of unbiased real Hadamard matrices. We will briefly discuss mutually unbiased bases in the last section. In the presentation of matrices we use j to denote $-i$ and $-$ to denote -1.

2 Unbiased complex Hadamard matrices

Dealing with complex matrices, i.e. matrices with entries in $\{-1,1,i,-i\}$, is quite different from working with the unimodular complex matrices as the powerful character theory is no longer applicable. We begin this section with a well known, but important property of complex Hadamard matrices.

Lemma 1. *Let $H=[h_{ij}]$ be a complex Hadamard matrix of order n for which the absolute values of the row sums are all identical and equal to $\mathbf{r}$. Then $\mathbf{r}=\sqrt{n}$.*

Proof. For $\mathbf{e}$ being the all ones vector, we have $(H\mathbf{e})^*(H\mathbf{e})=\mathbf{e}^*H^*H\mathbf{e}=\mathbf{e}^*nI\mathbf{e}=n\mathbf{e}^*\mathbf{e}=n^2$. So, $\sum_{i=1}^{n}|r_i|^2=n^2$, where $r_i=\sum_{j=1}^{n}h_{ij}$, $1\le i\le n$. It follows that $\mathbf{r}=\sqrt{n}$. □

A complex Hadamard matrix of order n for which the absolute values of the row sums are all equal to $\sqrt{n}$ is called *row regular*. It follows from Lemma 1 that for a row regular complex Hadamard matrix $H=[h_{kj}]$ of order $2n$, n odd, if $\sum_{j=1}^{2n}h_{kj}=a+ib$, for some k, $1\le k\le 2n$, then $a^2+b^2=2n$ and so both $|a|$ and $|b|$ are odd integers.

Lemma 2. *There is no pair of unbiased row regular complex Hadamard matrices of order $2n$, n odd.*

Proof. Suppose on the contrary that there is a pair of row regular complex Hadamard matrices H and K of order $2n$ such that $HK^*=L$, where the entries of L are of absolute value $\sqrt{2n}$. Let J be the matrix of all one entries of order $2n$. Then the matrix

$$\frac{1}{1+i}(H+J)\frac{1}{1+i}(K^*+J)$$

is a complex integer matrix (i.e. all entries of the matrix consist of Gaussian integers). To see this note that the entries of both matrices $\frac{1}{1+i}(H+J)$ and $\frac{1}{1+i}(K^*+J)$ belong to the set $\{0,1,-i,1-i\}$. Observing that

$$\frac{1}{1+i}(H+J)\frac{1}{1+i}(K^*+J)=\frac{-i}{2}(HK^*+HJ+JK^*+2nJ)$$

and that all the entries of the matrices HK^*, HJ and JK^* consist of numbers of the form $x+iy$, where both $|x|$ and $|y|$ are odd integers, we get a contradiction. □

Note that in the above proof we only use the fact that all the entries of the matrices HK^*, HJ and JK^* consist of numbers of the form $x+iy$, where both $|x|$ and $|y|$ are odd integers. So if there are two complex Hadamard matrices H, K of order $2n$, n odd, for which the row sums of H and K are all of the form $x+iy$, where both $|x|$ and $|y|$ are odd integers, then none of the entries of HK^* are of this form. Consequently, such H,K can not be unbiased, because otherwise the entries of $HK^* = [L_{ij}]$ would have to satisfy $|L_{ij}|^2 = 2n$, which must be a sum of two odd squares.

Theorem 3. *For any odd integer n,* $|\mathbf{MUCH}(2n)| \leq 2$.

Proof. Suppose on the contrary that there are more than two **MUCH** matrices of order $2n$. By multiplying the columns of all matrices by appropriate numbers we can make the first row of one of the matrices to be all equal to one. The new matrices form a set of **MUCH** matrices which contain at least two row regular Hadamard matrices of order $2n$, contradicting Lemma 1 and thus the result follows. □

Example 4. *Let*

$$H = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix}, \quad K = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Then

$$HK^* = \begin{pmatrix} 1-i & 1-i \\ i+1 & -i-1 \end{pmatrix}.$$

This shows the inequality in the Theorem 3 is sharp for $n = 1$.

We have conducted a computer search and found many maximal sets of **MUCH** matrices of orders 10 and 18. One representative from each of these pairs of matrices is listed below in Tables 1 and 2.

We believe that the upper bound in Theorem 3 is sharp for every odd integer n for which $2n$ is the order of a row regular complex Hadamard matrix. The following conjecture includes this and a conjecture regarding the existence of row regular complex Hadamard matrices.

Conjecture 5. $|\mathbf{MUCH}(2n)| = 2$ *for all odd integers n, where* $2n$ *is a sum of two squares.*

The existence of row regular Hadamard matrix is a necessary condition to have two **MUCH**'s (see the proof of Theorem 3). For matrices of size $2n$, n odd, the existence of a row regular Hadamard matrix is, in turn, conditioned by existence of integers a,b such that $2n = a^2+b^2$ (see lemma 1).

Table 1: A pair H, K of unbiased complex Hadamard matrices of order 10

$$
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1\\
1&1&-&-&-&i&j&1&-&1\\
1&-&1&-&-&j&-&i&1&1\\
1&-&-&1&-&j&1&1&i&-\\
1&-&-&-&1&i&1&-&1&j\\
1&j&i&i&j&-&i&j&j&i\\
1&i&-&1&1&j&-&-&-&1\\
1&1&j&1&-&i&-&-&1&-\\
1&-&1&j&1&i&-&1&-&-\\
1&1&1&-&i&j&1&-&-&-
\end{pmatrix},
\qquad
\begin{pmatrix}
j&-&-&1&1&1&1&j&i&1\\
j&1&1&1&i&-&j&1&i&j\\
1&j&i&j&1&-&1&i&1&i\\
1&i&j&1&j&j&-&1&1&i\\
i&1&-&i&j&i&1&1&1&j\\
i&1&i&j&1&1&-&1&-&1\\
1&i&1&1&1&i&i&-&j&j\\
-&j&1&1&-&1&i&i&1&1\\
1&1&j&-&i&1&j&-&1&1\\
-&1&1&i&1&j&1&j&j&i
\end{pmatrix}
$$

Table 2: A pair H, K of unbiased complex Hadamard matrices of order 18

$$
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&-&j&j&i&j&i&j&j&j&j&i&i&i&j&i&i&i\\
1&j&-&j&i&i&i&i&j&i&j&i&j&j&i&i&j&j\\
1&j&j&-&i&i&j&i&i&j&i&i&j&i&j&j&j&i\\
1&i&i&i&-&j&i&i&j&j&i&i&i&j&j&j&j&j\\
1&j&i&i&j&-&i&i&i&j&j&j&j&j&j&i&i&i\\
1&i&i&j&i&i&-&j&j&j&i&j&j&j&i&j&i&i\\
1&j&i&i&i&i&j&-&j&i&j&j&i&i&j&j&i&j\\
1&j&j&i&j&i&j&j&-&j&i&i&i&j&i&i&i&j\\
1&j&i&j&j&j&j&i&j&-&i&j&i&i&i&i&j&i\\
1&j&j&i&i&j&i&j&i&i&-&j&i&j&i&j&j&i\\
1&i&i&i&i&j&j&j&i&j&j&-&j&i&i&i&j&j\\
1&i&j&j&i&j&j&i&i&i&i&j&-&j&j&i&i&j\\
1&i&j&i&j&j&j&i&j&i&j&i&j&-&i&j&i&i\\
1&j&i&j&j&j&i&j&i&i&i&i&j&i&-&j&i&j\\
1&i&i&j&j&i&j&j&i&i&j&i&i&j&j&-&j&i\\
1&i&j&j&j&i&i&i&i&j&j&j&i&i&i&j&-&j\\
1&i&j&i&j&i&i&j&j&i&i&j&j&i&j&i&j&-
\end{pmatrix},
\qquad
\begin{pmatrix}
1&i&i&j&j&j&j&i&i&j&-&1&j&1&1&j&1&-\\
1&i&i&j&i&j&i&j&-&i&i&1&1&-&1&i&j&1\\
1&i&i&i&i&i&1&1&j&-&j&i&j&1&j&1&-&i\\
1&j&i&i&i&1&j&i&i&1&1&1&i&j&-&i&j&-\\
1&j&i&-&j&j&1&j&j&1&1&-&i&j&1&j&i&i\\
1&i&j&-&-&i&j&j&1&i&j&j&1&j&i&1&1&j\\
1&i&j&1&1&i&i&-&i&j&i&-&i&j&j&1&1&i\\
j&-&1&i&j&1&i&i&1&i&i&i&j&j&1&i&i&j\\
i&1&1&j&i&1&i&i&j&j&j&j&1&j&j&-&i&j\\
1&j&1&j&j&i&-&1&j&1&-&i&i&i&j&1&j&j\\
i&-&j&i&1&j&j&i&j&i&i&j&1&1&j&j&j&1\\
1&j&1&1&-&i&i&i&-&i&1&j&j&i&i&j&1&i\\
j&1&j&j&i&-&j&i&j&1&i&i&1&1&i&i&i&i\\
1&j&-&1&-&1&j&j&1&j&i&j&j&i&j&i&i&1\\
1&1&j&i&j&j&i&-&j&j&j&1&-&1&i&i&j&1\\
1&1&j&i&i&j&-&1&i&-&1&i&i&i&1&j&i&j\\
i&j&1&1&i&-&1&j&1&j&i&i&j&j&i&j&j&j\\
j&i&i&i&1&i&1&1&j&j&i&j&i&i&i&-&1&j
\end{pmatrix}
$$

3 Unbiased real Hadamard matrices

Two Hadamard matrices H, K of order n are called unbiased, if $HK^t = L$, where the absolute values of all entries of L are equal to $\sqrt{n}$. It follows that $L = \sqrt{n}A$, where A is a Hadamard matrix of order n. It is only recently that interest has been shown in unbiased Hadamard matrices

[1, 9] and some new applications have emerged [5]. Pairs of unbiased Hadamard matrices exist only in square orders, as $L = HK^t$, with moduli of entries of L equal to $\sqrt{n}$, is a matrix of integers. It is known and easy to prove (as shown below) that the maximum number of mutually unbiased Hadamard matrices of order $4n^2$, n odd, does not exceed 2. Although Lemma 13 provides an upper bound for the number of what we call weakly unbiased Hadamard matrices (see Definition. 7), unbiased Hadamard matrices of order $4n$, n an odd square, belong to this class (see Remark. 8). So Lemma 13 also applies to them. Until very recently no example for which the upper bound 2 is attained was known besides the trivial example of Hadamard matrices of order 4. The first non-trivial example of unbiased Hadamard matrices of order 36 is shown in [4]. The approach in [4] was to use a database of known Hadamard matrices of order 36 to search for matrices with unbiased mates. Interestingly, only a very small fraction of the over 3 million known matrices of order 36 which were tested had unbiased mates. In this section we show that some sets of **MUCH** matrices of order $2n$ can be used to generate sets of mutually unbiased Hadamard matrices of order $4n$. Having found pairs of **MUCH** of order 18, we have many pairs of mutually unbiased Hadamard matrices of order 36. We begin with a known [1] and simple lemma. Our motivation for including the proof here will follow.

Lemma 6. *There is no pair of unbiased row regular Hadamard matrices of order $4n^2$, n odd.*

Proof. Repeating the line of proof of Lemma 2, we have

$$\frac{1}{2}(H+J)\frac{1}{2}(K^t+J) = \frac{1}{4}(HK^t + HJ + JK^t + 4n^2J).$$

Noting that $HK^t = 2nL$, where L is a Hadamard matrix, we get a contradiction to the fact that the left side of the above identity is an integer matrix. □

A quick glance at the above proof reveals that $HJ + JK^t + 4n^2J \equiv 0 \pmod 4$, if and only if $HJ + JK^t \equiv 0 \pmod 4$. Assuming that $HJ + JK^t \equiv 0 \pmod 4$, we get a contradiction if we assume one (or equivalently all) of the entries of HK^t is equal to $2 \pmod 4$. This is our motivation for the following definition.

Definition 7. *Two Hadamard matrices H, K of order n are said to be weakly unbiased, if $|\{|a_{ij}| : 1 \le i \le n, 1 \le j \le n\}| \le 2$, and $HK^t = [a_{ij}] \equiv 2J \pmod 4$.*

Remark 8. *Note that for the unbiased Hadamard matrices H, K of order n, $|\{|a_{ij}| : 1 \le i \le n, 1 \le j \le n\}| = 1$, where $HK^t = [a_{ij}]$. So weakly unbiased Hadamard matrices are the natural extension of unbiased Hadamard matrices of order $4n$, n an odd square.*

The following lemma is immediate, using equality from the proof of Lemma 6.

Lemma 9. *Let H, K be Hadamard matrices of order $4n$ such that $HJ + JK^t \equiv 0 \pmod 4$. Then no entry of HK^t is equal to $2 \pmod 4$.*

Definition 10. *Two Hadamard matrices H, K of the same order are called to be modularly homogeneous if $HJ + JK^t \equiv 0 \pmod 4$.*

Lemma 11. *There is no pair H, K of modularly homogeneous Hadamard matrices of order $4n$ for which $HK^t \equiv 2J \pmod 4$.*

Proof. This follows from Lemma 9. $\square$

Remark 12. *The assumption that H and K are modularly homogeneous in Lemma 11 is essential. The Hadamard matrices of order 12 in Table 3 are weakly unbiased, that is $HK^t \equiv 2J$ $\pmod 4$, but not modularly homogeneous. It is noteworthy that the number of entries with value 2 or 6 in HK^t is not balanced as there are more 2 entries than 6 entries.*

Table 3: A pair H, K of weakly unbiased Hadamard matrices of order 12

$$\begin{pmatrix}
1&1&1&-&1&1&-&1&1&-&1&1\\
1&1&1&1&-&1&1&-&1&1&-&1\\
1&1&1&1&1&-&1&1&-&1&1&-\\
1&-&-&1&1&1&-&1&1&1&-&-\\
-&1&-&1&1&1&1&-&1&-&1&-\\
-&-&1&1&1&1&1&1&-&-&-&1\\
1&-&-&1&-&-&1&1&1&-&1&1\\
-&1&-&-&1&-&1&1&1&1&-&1\\
-&-&1&-&-&1&1&1&1&1&1&-\\
1&-&-&-&1&1&1&-&-&1&1&1\\
-&1&-&1&-&1&-&1&-&1&1&1\\
-&-&1&1&1&-&-&-&1&1&1&1
\end{pmatrix},
\quad
\begin{pmatrix}
-&1&1&-&1&1&1&-&-&1&1&1\\
1&-&1&1&-&1&-&1&-&1&1&1\\
1&1&-&1&1&-&-&-&1&1&1&1\\
1&-&-&-&1&1&1&1&1&-&1&1\\
-&1&-&1&-&1&1&1&1&1&-&1\\
-&-&1&1&1&-&1&1&1&1&1&-\\
-&1&1&-&-&-&-&1&1&-&1&1\\
1&-&1&-&-&-&1&-&1&1&-&1\\
1&1&-&-&-&-&1&1&-&1&1&-\\
-&-&-&1&-&-&1&-&-&-&1&1\\
-&-&-&-&1&-&-&1&-&1&-&1\\
-&-&-&-&-&1&-&-&1&1&1&-
\end{pmatrix}$$

Lemma 13. *Let $w(n)$ be the number of mutually weakly unbiased Hadamard matrices of order $4n$, n odd, then $w(n) \leq 2$.*

Proof. Suppose on the contrary that there are more than two mutually weakly unbiased Hadamard matrices of order $4n$. By negating the appropriate columns of all matrices, we may assume that one of the matrices has one normalized row. Select two other matrices, say H, K. Then $HJ + JK^t \equiv 0 \pmod 4$ and $HK^t \equiv 2J \pmod 4$, contradicting Lemma 9. $\square$

We are now ready for the main result of this section and our reason for studying unbiased complex Hadamard matrices. We need to introduce a notation first. For the integers a, b let $G(a,b) = \{a \pm ib, -a \pm ib, ia \pm b, -ia \pm b\}$.

Theorem 14. *Let H, K be a pair of unbiased complex Hadamard matrices of order $2n$, n odd, for which the entries of HK^* are all in $G(a,b)$, where $2n = a^2 + b^2$, a, b odd integers. Then there is a pair of weakly unbiased Hadamard matrices of order $4n$.*

Proof. Let $H = A + iB$, $K = C + iD$, where A, B and C, D are $(0, \pm 1)$-matrices of order $2n$ such that $A \pm B$ and $C \pm D$ are ± 1-matrices. Consider the matrices

$$H' = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes A + \begin{pmatrix} - & 1 \\ 1 & 1 \end{pmatrix} \otimes B$$

and

$$K' = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix} \otimes C + \begin{pmatrix} - & 1 \\ 1 & 1 \end{pmatrix} \otimes D.$$

It is only a routine calculation to see that H', K' are Hadamard matrices of order $4n$. Let $HK^* = E + iF$, where E, F are $(\pm a, \pm b)$-matrices of order $2n$. We have

$$H'K'^t = \begin{pmatrix} 2(AC^t + BD^t) & -2(BC^t - AD^t) \\ 2(BC^t - AD^t) & 2(AC^t + BD^t) \end{pmatrix} = \begin{pmatrix} 2E & -2F \\ 2F & 2E \end{pmatrix}.$$

Using the fact that the entries of HK^* are in $G(a, b)$ and noting that E, F are $(\pm a, \pm b)$-matrices, where $|a|$,$|b|$ are odd integers, it follows that H', K' are weakly unbiased. □

Remark 15. *The spread of a's and b's in $H'K'^t$ is uniform; there are as many a's in $H'K'^t$ as b's. We think the assumption that all the entries of HK^* belong to $G(a, b)$ is not necessary, but we cannot prove it.*

Theorem 16. *Let H, K be a pair of unbiased complex Hadamard matrices of order $2n$, where $n = a^2$, a odd (and so $2n = a^2 + a^2$) for which the entries of HK^* are in $G(a, a)$. Then H', K' constructed above form a pair of unbiased Hadamard matrices of order $4n = 4a^2$.*

Proof. Note that in this case the matrices E and F in the proof of Theorem 14 are both $\pm a$-matrices. □

Corollary 17. *There is a pair of unbiased Hadamard matrices of order* 36.

Proof. We apply Theorem 16 to the pair of unbiased complex Hadamard matrices of order 18 of Table 2. The resulting pair of matrices is given in Tables 4 and 5. The fact that all entries of HK^* are in $G(a, a)$ is automatic in this case, as 18 is sum of two squares in only one way. □

Corollary 18. *There is a pair of weakly unbiased Hadamard matrices of order* 20.

Proof. We apply Theorem 14 to the pair of unbiased complex Hadamard matrices of order 10 of Table 1. The resulting pair of matrices is given in Table 6. All entries of HK^* are in $G(a, b)$, where $\{a, b\} = \{1, 3\}$. □

Consider the even integer $2n$, $n = a^2$ for some odd integer a, and assume that $2n = a^2 + a^2$ is the only way that $2n$ can be written as sum of two squares. Let H, K be two unbiased complex Hadamard matrices H, K of order $2n$. It is easy to see that $HK^* = (a + ia)L$, where L is a complex Hadamard matrix of order $2n$. A pair of unbiased complex Hadamard matrices H, K of order $2n$, $2n = a^2 + b^2$, is called *special* if $HK^* = (a + ib)L$ for some complex Hadamard matrix L. The unbiased complex Hadamard matrices of orders 2 and 18 above are special. We did an exhaustive computer search and found none of order 10.

Table 4: A pair of unbiased Hadamard matrices of order 36: first matrix

$$H = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
1 & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 \\
1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 \\
1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - \\
1 & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - \\
1 & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 \\
1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 \\
1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - \\
1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - \\
1 & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - \\
1 & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - \\
1 & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - \\
1 & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 \\
1 & - & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 \\
1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 \\
1 & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 \\
1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - \\
1 & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - \\
1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - \\
1 & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - \\
1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - \\
1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - & - & 1 \\
1 & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & 1 \\
1 & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 & - \\
1 & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & - \\
1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & 1 \\
1 & - & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 \\
1 & - & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & 1 \\
1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - \\
1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1
\end{pmatrix}$$

Table 5: A pair of unbiased Hadamard matrices of order 36: second matrix

$$K = \begin{pmatrix}
1&1&1&-&1&-&-&1&1&-&-&1&1&-&1&-&-&-&1&-&-&1&1&1&1&1&1&1&1&1&-&1&1&-&-&- \\
1&-&-&-&-&-&1&1&-&-&1&1&-&-&-&-&-&1&-&-&1&1&1&-&1&-&1&-&1&-&1&1&-&-&-&1 \\
1&1&1&-&-&1&1&-&-&1&1&-&-&1&1&-&1&1&-&-&-&1&1&1&-&1&-&1&1&1&-&1&-&-&1&1 \\
1&-&-&-&1&1&-&-&1&1&-&-&1&1&-&-&1&-&-&1&1&1&1&-&1&1&1&1&1&-&1&1&-&1&1&- \\
1&1&1&-&-&1&-&1&1&-&1&-&1&1&1&1&-&1&-&-&1&-&1&-&1&-&-&-&-&1&1&1&1&1&1&- \\
1&-&-&-&1&1&1&1&-&-&-&-&1&-&1&-&1&1&-&1&-&-&-&-&-&-&-&1&1&1&1&-&1&-&-&- \\
1&1&1&-&-&1&1&-&1&-&-&-&1&-&-&1&1&1&1&1&1&1&1&1&-&1&1&-&-&-&1&-&1&-&-&1 \\
1&-&-&-&1&1&-&-&-&-&-&1&-&-&1&1&1&-&1&-&1&-&1&-&1&1&-&-&-&1&-&-&-&-&1&1 \\
1&1&1&-&1&-&1&1&-&1&-&1&-&1&-&1&1&1&1&-&1&-&-&-&-&1&1&1&-&1&1&1&-&1&-&- \\
1&-&-&-&-&-&1&-&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-&1&1&1&1&-&1&1&1&-&1&1&-&1 \\
1&1&-&1&-&1&-&-&-&-&-&1&-&1&-&-&1&1&-&-&-&-&1&1&1&-&1&1&-&-&-&-&1&1&-&- \\
1&-&1&1&1&1&-&1&-&1&1&1&1&1&-&1&1&-&-&1&-&1&1&-&-&-&1&-&-&1&-&1&1&-&-&1 \\
1&1&-&1&1&-&1&1&-&-&1&-&-&-&1&1&1&-&-&1&1&-&1&1&-&1&1&-&1&-&-&1&1&1&1&- \\
1&-&1&1&-&-&1&-&-&1&-&-&-&1&1&-&-&-&1&1&-&-&1&-&1&1&-&-&-&-&1&1&1&-&-&- \\
1&1&-&1&1&-&-&-&1&1&1&1&-&-&1&-&-&1&-&1&1&1&1&-&-&1&-&1&-&1&1&-&1&1&-&1 \\
1&-&1&1&-&-&-&1&1&-&1&-&-&1&-&-&1&1&1&1&1&-&-&-&1&1&1&1&1&1&-&-&1&-&1&1 \\
1&1&1&-&1&-&1&1&1&1&-&-&-&-&-&-&1&-&-&-&-&1&-&-&1&-&-&-&-&-&-&-&1&1&1&1 \\
1&-&-&-&-&-&1&-&1&-&-&1&-&1&-&1&-&-&-&1&1&1&-&1&-&-&-&1&-&1&-&1&1&-&1&- \\
1&1&-&1&1&1&1&-&-&1&1&-&1&-&-&-&-&-&1&-&1&-&-&1&1&-&1&1&-&1&1&1&1&-&1&1 \\
1&-&1&1&1&-&-&-&1&1&-&-&-&-&-&1&-&1&-&-&-&-&1&1&-&-&1&-&1&1&1&-&-&-&1&- \\
1&1&-&1&1&1&1&-&1&-&-&-&-&1&1&1&-&1&1&1&-&1&-&-&1&-&1&-&1&1&-&1&-&1&-&1 \\
1&-&1&1&1&-&-&-&-&-&-&1&1&1&1&-&1&1&1&-&1&1&-&1&-&-&-&-&1&-&1&1&1&1&1&1 \\
1&1&-&1&-&-&-&1&-&-&-&-&1&1&1&1&-&-&-&-&-&1&-&-&-&1&1&1&-&-&1&-&-&-&1&1 \\
1&-&1&1&-&1&1&1&-&1&-&1&1&-&1&-&-&1&-&1&1&1&-&1&1&1&1&-&-&1&-&-&-&1&1&- \\
1&1&1&1&-&1&-&1&1&1&-&1&-&-&1&1&1&-&-&1&1&-&-&1&1&-&-&1&1&-&1&1&-&-&-&1 \\
1&-&1&-&1&1&1&1&1&-&1&1&-&1&1&-&-&-&1&1&-&-&1&1&-&-&1&1&-&-&1&-&-&1&1&1 \\
1&1&-&-&1&1&-&1&-&1&1&1&-&1&-&1&-&1&1&1&-&1&-&1&1&1&-&-&1&-&1&-&1&-&1&- \\
1&-&-&1&1&-&1&1&1&1&1&-&1&1&1&1&1&1&1&-&1&1&1&1&1&-&-&1&-&-&-&-&-&-&-&- \\
1&1&1&1&1&1&1&-&1&-&1&1&1&1&-&-&-&-&-&-&1&-&-&-&-&1&-&-&1&-&-&-&-&-&-&- \\
1&-&1&-&1&-&-&-&-&-&1&-&1&-&-&1&-&1&-&1&-&-&-&1&1&1&-&1&-&-&-&1&-&1&-&1 \\
1&1&-&-&-&-&-&1&-&1&-&-&1&1&-&-&-&-&1&1&1&-&1&1&-&-&-&-&1&1&-&-&-&1&-&1 \\
1&-&-&1&-&1&1&1&1&1&-&1&1&-&-&1&-&1&1&-&-&-&1&-&-&1&-&1&1&-&-&1&1&1&1&1 \\
1&1&1&1&-&-&1&-&-&-&1&1&1&-&-&1&1&-&1&1&-&1&1&-&1&-&-&1&1&1&1&-&-&1&1&- \\
1&-&1&-&-&1&-&-&-&1&1&-&-&-&1&1&-&-&1&-&1&1&-&-&-&-&1&1&1&-&-&-&1&1&-&- \\
1&1&-&-&-&-&-&-&1&1&1&1&1&-&1&-&1&1&1&1&-&-&-&-&-&-&1&-&-&-&-&1&-&-&1&- \\
1&-&-&1&-&1&-&1&1&-&1&-&-&-&-&-&1&-&1&-&-&1&-&1&-&1&-&-&-&1&1&1&-&1&-&-
\end{pmatrix}$$

Table 6: A pair H, K of weakly unbiased Hadamard matrices of order 20

$$
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&-&1&-&1&-&1&-&1&-&1&-&1&-&1&-&1&-&1&-\\
1&1&-&-&-&1&-&1&1&-&-&1&1&-&1&-&1&-&-&1\\
1&-&-&1&1&1&1&1&-&-&1&1&-&-&-&-&-&-&1&1\\
1&1&-&1&-&-&-&1&1&-&1&-&-&1&-&1&1&-&1&-\\
1&-&1&1&-&1&1&1&-&-&-&-&1&1&1&1&-&-&-&-\\
1&1&-&1&-&1&-&-&-&1&1&-&1&-&1&-&-&1&1&-\\
1&-&1&1&1&1&-&1&1&1&-&-&-&-&-&-&1&1&-&-\\
1&1&1&-&1&-&-&1&-&-&-&1&-&1&1&-&-&1&1&-\\
1&-&-&-&-&-&1&1&-&1&1&1&1&1&-&-&1&1&-&-\\
1&1&-&1&1&-&1&-&-&1&-&-&-&1&1&-&1&-&-&1\\
1&-&1&1&-&-&-&-&1&1&-&1&1&1&-&-&-&-&1&1\\
1&1&1&-&-&1&1&-&-&1&-&1&-&-&-&1&1&-&1&-\\
1&-&-&-&1&1&-&-&1&1&1&1&-&1&1&1&-&-&-&-\\
1&1&1&-&-&1&1&-&1&-&1&-&-&1&-&-&-&1&-&1\\
1&-&-&-&1&1&-&-&-&-&-&-&1&1&-&1&1&1&1&1\\
1&1&1&-&1&-&-&1&-&1&1&-&1&-&-&1&-&-&-&1\\
1&-&-&-&-&-&1&1&1&1&-&-&-&-&1&1&-&1&1&1\\
1&1&-&1&1&-&1&-&1&-&-&1&1&-&-&1&-&1&-&-\\
1&-&1&1&-&-&-&-&-&-&1&1&-&-&1&1&1&1&-&1
\end{pmatrix},
\begin{pmatrix}
1&1&1&-&-&1&-&1&-&1&1&1&-&-&-&-&-&1&1&1\\
1&-&-&-&1&1&1&1&1&1&1&-&-&1&-&1&1&1&1&-\\
1&1&1&-&1&1&1&-&1&-&-&1&-&-&1&-&1&1&-&-\\
1&-&-&-&1&-&-&-&-&-&1&1&-&1&-&-&1&-&-&1\\
1&1&-&1&-&-&1&-&-&-&-&-&-&1&-&-&-&1&1&-\\
1&-&1&1&-&1&-&-&-&1&-&1&1&1&-&1&1&1&-&-\\
1&1&-&1&1&1&1&1&-&1&1&1&1&1&1&-&-&-&-&-\\
1&-&1&1&1&-&1&-&1&1&1&-&1&-&-&-&-&1&-&1\\
1&1&-&1&-&-&1&1&1&1&-&1&-&-&-&1&1&-&-&1\\
1&-&1&1&-&1&1&-&1&-&1&1&-&1&1&1&-&-&1&1\\
1&1&1&-&-&-&1&1&-&-&1&-&1&1&1&1&1&1&-&1\\
1&-&-&-&-&1&1&-&-&1&-&-&1&-&1&-&1&-&1&1\\
1&1&-&-&-&1&-&-&1&-&1&-&1&-&-&1&-&-&-&-\\
1&-&-&1&1&1&-&1&-&-&-&-&-&-&1&1&-&1&-&1\\
1&1&1&1&1&-&-&-&-&1&1&-&-&-&1&1&1&-&1&-\\
1&-&1&-&-&-&-&1&1&1&-&-&-&1&1&-&-&-&-&-\\
1&1&-&-&1&-&-&-&1&1&-&1&1&1&1&1&-&1&1&1\\
1&-&-&1&-&-&-&1&1&-&1&1&1&-&1&-&1&1&1&-\\
1&1&1&1&1&1&-&1&1&-&-&-&1&1&-&-&1&-&1&1\\
1&-&1&-&1&-&1&1&-&-&-&1&1&-&-&1&-&-&1&-
\end{pmatrix}
$$

4 Unbiased bases

Let H, K be a pair of special unbiased complex Hadamard matrices of order $2n^2$ corresponding to the decomposition $2n^2 = n^2 + n^2$, so that $HK^* = (n+in)L$, for some complex Hadamard matrix L. Then the normalized rows of H and K, or equivalently the rows of $\frac{1}{\sqrt{2n^2}}H$ and $\frac{1}{\sqrt{2n^2}}K$, form two orthonormal bases for $\mathbb{C}^{2n^2}$ in such a way that for every pair of vectors u, v from different bases, $\langle u, v\rangle \in \mathcal{D} = \{\frac{1}{2n}(1+i), -\frac{1}{2n}(1+i), \frac{1}{2n}(1-i), -\frac{1}{2n}(1-i)\}$ (note that $\frac{n+in}{2n^2} = \frac{1}{2n}(1+i)$). Here $\langle,\rangle$ denotes the standard Hermitian inner product in $\mathbb{C}^{2n^2}$. Adding $\{\frac{1+i}{\sqrt{2}}\,\mathbf{b} : \mathbf{b} \in B_s\}$, where B_s denotes the standard basis in $\mathbb{C}^{2n^2}$, to these bases we get 3 orthonormal bases for $\mathbb{C}^{2n^2}$ in such a way that for every pair of vectors u, v from different bases, $\langle u, v\rangle \in \mathcal{D}$. Two orthonormal bases $\mathcal{B}_1$ and $\mathcal{B}_2$ in $\mathbb{C}^{2n^2}$ are called *unbiased complex bases* if $\langle u, v\rangle \in \mathcal{D}$ for all $u \in \mathcal{B}_1$ and $v \in \mathcal{B}_2$.

We will use $|\mathbf{MUCB}(n)|$ to denote the number of elements in a set of mutually unbiased complex bases for $\mathbb{C}^n$.

Lemma 19. $|\mathbf{MUCB}(2n^2)| \leq 3$ *for any odd integer n. Equality is attained for* $n = 1, 3$.

Proof. Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ be three mutually unbiased complex bases for $\mathbb{C}^{2n^2}$. Let H_i be the matrix formed by putting the vectors of $\mathcal{B}_i$ as the rows of H_i, $i = 1, 2, 3$. Then $\frac{2n}{1+i} H_2 H_1^*$ and $\frac{2n}{1+i} H_3 H_1^*$ form a special pair of unbiased complex Hadamard matrices of order $2n^2$ corresponding to the decomposition $2n^2 = n^2 + n^2$. Thus, it follows from Theorem 3 that $|\mathbf{MUCB}(2n^2)| - 1 \leq 2$. The equality occurs for $n = 1, 3$ as there are pair of special unbiased complex Hadamard matrices of order 2 and 18. □

Two orthonormal bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for $\mathbb{R}^n$ are called *mutually unbiased real bases* if $\langle u, v \rangle \in \{\frac{1}{\sqrt{n}}, -\frac{1}{\sqrt{n}}\}$ for all $u \in \mathcal{B}_1$ and $v \in \mathcal{B}_2$, where $\langle , \rangle$ is the standard Euclidean inner product in $\mathbb{R}^n$, see [1] for details. We will use $|\mathbf{MURB}(n)|$ to denote the number of elements in a set of mutually unbiased real bases in $\mathbb{R}^n$.

Lemma 20. $|\mathbf{MURB}(4n^2)| \leq 3$ *for any odd integer n. Equality is attained for* $n = 1, 3$.

Proof. Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ be three mutually unbiased real bases for $\mathbb{R}^{4n^2}$. Let H_i be the matrix formed by putting the vectors of $\mathcal{B}_i$ as the rows of H_i, $i = 1, 2, 3$. Then $2nH_2H_1^t$ and $2nH_3H_1^t$ form a pair of unbiased Hadamard matrices of order $4n^2$. The result now follows from Lemma 13 and Corollary 17. See also Observation 2.1 of [1]. □

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