Third Combinatorics Day

University of Lethbridge

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Three problems between graph theory and topology

Richard K. Guy

1. Lovasz proved a conjecture of Kneser that the $\binom{n}{r}$ r-tuples chosen from an n-set may be partitioned into $n-2r+2$ parts, so that every pair in the same part have a non-empty intersection. For example, the edges of K_6 may be partitioned into 4 parts:

01, 02, 03, 04, 05; 12, 13, 14, 15; 23, 24, 25; 34, 35, 45.

Under what circumstances can the cardinalities of the parts be equal? (Clearly not in the present example.) Under what circumstances can the parts, considered as (hyper)graphs, be isomorphic?

2. Show that the outerthickness of the complete graph on $4n-1$ points, K_{4n-1}, is n (provided $n \neq 2$). That is, show that its $(2n-1)(4n-1)$ edges can be partitioned into n parts, each of which can be drawn as a planar graph with all its vertices on the boundary of a single cell.

3. Find the outercoarseness of the (edge-set of the) 5-dimensional cube, Q_5. I.e., partition its $5 \times 2^{5-1} = 80$ edges into as many non-outerplanar graphs as possible. The answer is either 7, 8 or 9. Obtain a better bound than

$$\xi_o(Q_n) > (0.96n - 1.15)2^{n-4}$$

On general partition graphs

Ton Kloks

A graph G is a general partition graph if there is some set S and an assignment of non–empty subsets $S_x \subseteq S$ to the vertices of G such that two vertices x and y are adjacent if and only if $S_x \cap S_y \neq \emptyset$ and for every maximal independent set M the set $\{S_m \mid m \in M\}$ is a partition of S. For every minor closed family of graphs there exists a polynomial time algorithm that checks if an element of the family is a general partition graph. Also for the class of circle graphs we show that it can be checked in polynomial time if a member of the class is a general partition graph.

The triangle condition says that for every maximal independent set M and for every edge (x,y) with $x,y \notin M$ there is a vertex $m \in M$ such that $\{x,y,m\}$ is a triangle in G. It is known that the triangle condition is necessary for a graph to be a general partition graph (but in general not sufficient). We show that for AT–free graphs this condition is also sufficient and this leads to an efficient algorithm that demonstrates whether or not an AT–free graph is a general partition graph.

We show that the triangle condition can be checked in polynomial time for planar graphs and circle graphs. It is unknown if the triangle condition is also a sufficient condition for planar graphs to be a general partition graph. For circle graphs we show that the triangle condition is *not* sufficient.

This covers joint work with C. M.Lee (Chung-Cheng University, Chiayi, Taiwan), Jim Liu (University of Lethbridge, Lethbridge, Canada), and Haiko Müller (School of Computing, Leeds, UK).

Hamiltonian properties of complements of line graphs

Jim Liu

Let $G = (V, E)$ be a simple graph. The complement of the line graph of G, denoted by $\overline{L(G)}$, has vertex set E, two vertices e_1 and e_2 are adjacent in $\overline{L(G)}$ if e_1 and e_2 are not incident in G. Let P be any of the properties: Hamiltonian, traceable, Hamilton-connected, Hamilton-laceable, and pancyclic. I will characterize graphs such that the complements of their line graphs have property P.

Moreover, these characterizations lead to linear recognition algorithms.

How to exchange secrets — communication of cryptographic keys

Renate Scheidler

Conventional (one-key) cryptographic systems, such as the Data Encryption Standard (DES) or the new Advanced Encryption Standard (AES), are the preferred secure communication schemes for many applications. This is because they are both fast and sufficiently secure for most applications. The real difficulty in employing such cryptosystems is the problem of securely transmitting a secret cryptographic key between communicants. This talk describes a solution to this problem – a means by which a secret key can be safely transmitted across an insecure channel. Our key exchange protocols are based on the algebra and arithmetic of reduced principal ideals in a real quadratic number field.

This research was conducted in collaboration with H. C. Williams and M. J. Jacobson, Jr., both at the University of Calgary, and J. A. Buchmann at the Technical University of Darmstadt, Germany.

Despite the quite algebraic and number theoretic nature of this topic, this presentation is designed to be accessible to a general mathematics-trained audience.

Stitching Images Back Together

Joy Morris

In a variety of settings, images may be broken down into smaller pieces. During the course of this, information can be lost or slightly distorted, creating problems when attempts are made to reconstruct the original image from the pieces. This talk discusses methods for reconstructing the original image, focusing on a graph-theoretic model for the problem.

Hamiltonian paths in cartesian powers of directed cycles

Dave Witte

The vertex set of the k^{th} cartesian power of a directed cycle of length m can be naturally identified with the abelian group $(\mathbb{Z}_m)^k$. For any two elements $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ of $(\mathbb{Z}_m)^k$, it is easy to see that if there is a hamiltonian path from u to v, then

$$u_1 + \cdots + u_k \equiv v_1 + \cdots + v_k + 1 \pmod{m}.$$

We prove the converse, unless $k = 2$ and m is odd. This is joint work with David Austin and Heather Gavlas. A similar result is conjectured for cartesian products of directed cycles that are not assumed to be of equal lengths.

Vertex Embeddings of Regular Polytopes

Peter Zvengrowski

Starting from the fact, known since antiquity, that it is possible to choose four vertices of the cube so as to form the vertices of a regular tetrahedron, we investigate in this mainly expository talk the general question of when the vertices of one regular polytope embed in those of another regular polytope. Relationships of this question with several areas of mathematics will be discussed, including combinatorics, linear algebra, number theory, Galois theory, and algebraic topology. The proof of the equivalence of the following three statements will be outlined:

(1) the regular $(n-1)$-simplex has a vertex embedding in the $(n-1)$-cube,

(2) there exists a Hadamard matrix of order n,

(3) the regular n-orthoplex (generalized octahedron) has a vertex embedding in the n-cube.