Congruence

Consider real symmetric matrices of size $n$ by $n$ only (these are orthogonal diagonalizable). $n$ is the order of the matrix.

If $B = P^T A P$ where $P$ is invertible then $A$ and $B$ are said to be congruent.

Under similarity, the eigenvalues stay the same, since the characteristic equation is same.

Question: Under congruence, what stays the same?

For $c_i \neq 0$ we have congruence

$$
\begin{pmatrix}
    c_1 & 0 & \cdots & 0 \\
    0 & c_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & c_n
\end{pmatrix}
\begin{pmatrix}
    \lambda_1 & 0 & \cdots & 0 \\
    0 & \lambda_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & \lambda_n
\end{pmatrix}
\begin{pmatrix}
    c_1 & 0 & \cdots & 0 \\
    0 & c_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & c_n
\end{pmatrix} =
\begin{pmatrix}
    c_1^2 \lambda_1 & 0 & \cdots & 0 \\
    0 & c_2^2 \lambda_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & c_n^2 \lambda_n
\end{pmatrix}
$$

so we can change the value of each eigenvalue to one of $+1$, $-1$ or $0$. Note permutation matrices are orthogonal, so we can change the order of the eigenvalues. Let $p$ be the number of $+1$’s, $m$ be the number of $-1$’s and $z$ be the number of $0$’s. Note $p + m + z = n$, the order. The integer $n - z$ is the rank and $p - m$ is called the signature. From the order, rank and signature we can recover $p$, $m$ and $z$.

Computational note: we can obtain the diagonal matrix above by using simultaneous row and column operations in a manner like ordinary row reduction — consider what $EAE^T$ means for elementary $E$. This avoids having to compute eigenvalues.

In the complex case, where Hermitian matrices are used instead, $\lambda_i$ is real. Use $P^*$ instead of $P^T$. Then $c_i \lambda_i c_i = |c_i|^2 \lambda_i$ so again the eigenvalues can be changed to one of $+1$, $-1$ or $0$ and similar results hold.

**Theorem** (Sylvester’s law of inertia) Symmetric matrices are congruent iff they have the same order, rank and signature (or equivalently, the same $p$, $m$ and $z$).

**Proof:** Let

$$
\Lambda_{p,m,z} = \text{Diagonal } \begin{pmatrix} \underbrace{p}_{1}, \underbrace{m}_{1}, \underbrace{z}_{0} \end{pmatrix}.
$$

It suffices to show $\Lambda_{p,m,z}$ is not congruent to any other $\Lambda_{p',m',z'}$. For this the case $p < p'$ is like all other cases. Suppose $\Lambda_{p,m,z} = P^T \Lambda_{p',m',z'} P$. Consider subspaces, where the $x_i$’s are arbitrary,

$$
V_1 = \{ P^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_{p'} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}, \quad V_{-1} = \{ \begin{pmatrix} : \\ 0 \\ x_{p+1} \\ : \\ \vdots \\ 0 \end{pmatrix} \}, \quad \text{and} \quad V_0 = \{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{p+m+1} \\ \vdots \\ x_{p+m+z} \end{pmatrix} \}
$$

Since $\dagger \ v^T \Lambda_{p,m,z} v$ is $>0$, $<0$ or $=0$ for all nonzero $v \in V_j$, as $j$ is 1, $-1$ or 0, we see that the $V_j$ are disjoint subspaces of $n$-space. Yet their dimensions add to $p' + m + z$ which is greater than $n$.

---

$\dagger$ Let $v = P^{-1} \begin{pmatrix} x \\ 0 \end{pmatrix}$ be in $V_1$. Then $v^T \Lambda_{p,m,z} v = (x \ 0) (P^{-1})^T \Lambda_{p,m,z} P^{-1} (x \ 0) = (x \ 0)^T \Lambda_{p',m',z'} (x \ 0) = x_1^2 + \cdots + x_{p'}^2 > 0$ if $v \neq 0$. 
