A Note on Derangements

There is a somewhat complicated formula obtained via inclusion-exclusion for the number d_n of derangements of 1, 2, ... n.

Alternately it is possible to derive the recurrence relation

$$d_n = (n-1)(d_{n-1} + d_{n-2})$$

with $d_2 = 1$, $d_1 = 0$ or perhaps better, with $d_1 = 0$, $d_0 = 1$.

This recurrence relations can be rewritten in a better form.
Bringing nd_{n-1} to the left gives:

$$d_n - nd_{n-1} = -d_{n-1} + (n-1)d_{n-2}.$$

Multiply by $(-1)^n$ gives:

$$(-1)^n(d_n - nd_{n-1}) = (-1)^{n-1}(d_{n-1} - (n-1)d_{n-2}).$$

Now iterate this formula, so that ultimately on the right when n is reduced to 3 we obtain:

$$= (-1)^2(d_2 - (2)d_1) = 1.$$

Thus:

$$(-1)^n(d_n - nd_{n-1}) = 1$$

Solving for d_n gives a recurrence relation, which is much better than the first one above:

$$d_n = nd_{n-1} + (-1)^n$$

with $d_1 = 0$ or better yet with $d_0 = 1$.

This recurrence is easy to implement and is strikingly similar to the one for $f_n = n!$, namely:

$$f_n = nf_{n-1}, \qquad f_0 = 1$$