Differentiation Definitions and Proofs

1. Newton Quotient Definition

The function f is differentiable at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. The value of the limit is denoted $(Df)(x)$.
Remark: The quotient is the slope of a secant line; the limit is the slope of the tangent line at $(x, f(x))$.

2. Tangent Line or Taylor Definition

The function f is differentiable at x if there is a number denoted $(Df)(x)$ such that

$$f(x+h) = f(x) + Df(x) \cdot h + R_f(x,h) \cdot h \qquad \text{where } R_f(x,h) \to 0 \text{ as } h \to 0.$$

Remark: Roughly this means $f(x+h)$ can be written as a constant part, a linear part and a nonlinear part, and the derivative exists if the nonlinear part becomes insignificant for small h. Solving for $R_f(x,h)$ gives $R_f(x,h) = \dfrac{f(x+h) - f(x)}{h} - Df(x)$ which shows the connection between the two definitions. The first definition is more useful for computing derivatives of specific functions. The second is useful for approximating functions and for theoretical purposes.

3. Proof of Differentiability implies Continuity

Suppose f is differentiable at x. Then let $h \to 0$ in the Taylor form of the derivative. We get immediately that $\lim_{h \to 0} f(x+h) = f(x)$. Thus f is continuous at x.

4. Proof of the Linearity and Product Rules

Suppose f and g are differentiable at x, and let a and b be constant.
Then there are numbers $Df(x)$ and $Dg(x)$ with

$$\begin{aligned} f(x+h) &= f(x) + Df(x)h + R_f h \qquad &&\text{where } R_f = R_f(x,h) \to 0 \text{ as } h \to 0, \text{ and} \\ g(x+h) &= g(x) + Dg(x)h + R_g h \qquad &&\text{where } R_g = R_g(x,h) \to 0 \text{ as } h \to 0. \end{aligned}$$

LINEARITY RULE: Multiply the first equation by a and the second by b and then add them to obtain, on rearranging

$$af(x+h) + bg(x+h) = af(x) + bg(x) + [aDf(x) + bDg(x)]\,h + [aR_f + bR_g]\,h$$

Since $aR_f + bR_g \to 0$ as $h \to 0$, it follows that $D(af + bg)(x) = aDf(x) + bDg(x)$.

PRODUCT RULE: Multiply the two equations together to get

$$
\begin{aligned}
f(x+h)g(x+h) &= [f(x)+Df(x)h+R_f h]\cdot[g(x)+Dg(x)h+R_g h] \\
&= f(x)g(x)+Df(x)hg(x)+R_f hg(x) \\
&\quad + f(x)Dg(x)h+Df(x)hDg(x)h+R_f hDg(x)h \\
&\quad + f(x)R_g h+Df(x)hR_g h+R_f hR_g h \\
&= f(x)g(x)+[Df(x)g(x)+f(x)Dg(x)]\,h+Rh \qquad (1)
\end{aligned}
$$

where

$$R = R_f g(x)+Df(x)hDg(x)+R_f hDg(x)+f(x)R_g+Df(x)hR_g+R_f hR_g$$

Now as $h \to 0$ we have $R \to 0$ since each term has one or more of h, R_f, or R_g each of which $\to 0$ while the rest of the term is constant with respect to h. Thus the linear term of the equation (1) gives $D(fg)(x) = Df(x)g(x)+f(x)Dg(x)$.

5. Proof of the Sum Rule

Use $a = b = 1$ in the linearity rule.

6. Proof of the Chain Rule

Suppose f is differentiable at $u = g(x)$ and g is differentiable at x. There is a number $Dg(x)$ with

$$g(x+h) = g(x)+Dg(x)h+R_g h \qquad \text{where } R_g = R_g(x,h) \to 0 \text{ as } h \to 0.$$

We need to expand $f(g(x+h))$ so applying f on the above would give

$$f(g(x+h)) = f(g(x)+Dg(x)h+R_g h)$$

but we want to get $f(g(x))$ or $f(u)$ out of this! Let $l = Dg(x)h+R_g h$. *Then the right side above becomes* $f(u+l)$. Since f is differentiable at $u = g(x)$ there is a number $Df(u)$ with

$$f(u+l) = f(u)+Df(u)l+R_f l \qquad \text{where } R_f = R_f(u,l) \to 0 \text{ as } l \to 0.$$

Note that $l \to 0$ whenever $h \to 0$.
Thus

$$
\begin{aligned}
f(g(x+h)) &= f(g(x)+Dg(x)h+R_g h) \\
&= f(u+l) \\
&= f(u)+Df(u)l+R_f l \\
\text{substitute for } u \text{ and } l: \quad &= f(g(x))+Df(g(x))\,[Dg(x)h+R_g h]+R_f\,[Dg(x)h+R_g h] \\
&= f(g(x))+Df(g(x))Dg(x)h+Rh \qquad (2)
\end{aligned}
$$

where

$$R = Df(g(x))R_g+R_f Dg(x)+R_f R_g$$

which $\to 0$ as $h \to 0$ since then also $l \to 0$ and so the R_g and R_f terms $\to 0$. Thus the linear term of the equation (2) gives $D(f \circ g)(x) = (Df)(g(x)) \cdot (Dg)(x)$.

7. Proof of the Reciprocal Rule

$D(1/f) = Df^{-1} = -f^{-2}Df$ using the chain rule and $Dx^{-1} = -x^{-2}$ in the last step.

8. Proof of the Quotient Rule

$D(f/g) = D(f \cdot g^{-1})$. Now use the product rule to get $Df \cdot g^{-1} + f \cdot D(g^{-1})$. Then use the same steps as in the reciprocal rule, that is, use the chain rule and $Dx^{-1} = -x^{-2}$ to get $Df \cdot g^{-1} - f \cdot g^{-2} \cdot Dg$, which is $g^{-2}[Df \cdot g - f \cdot Dg]$.

9. Proof of the Inverse Rule

If $y = f(x)$ and $x = g(y)$ then $g(f(x)) = x$ so differentiating with respect to variable x and using the chain rule on the left gives $\frac{dg(f(x))}{df(x)}\frac{df(x)}{dx} = 1$. Thus $\frac{dg(y)}{dy}\frac{df(x)}{dx} = 1$ and so $\frac{dg(y)}{dy} = 1/\frac{df(x)}{dx}$.

10. Proof of Logarithmic Differentiation

Use the chain rule to get

$$f \cdot D(\ln|f|) = f \cdot \frac{1}{f} \cdot Df = Df.$$

11. Proof of the Fundamental Theorem

By definition $A(x) = \int_a^x f(t)\,dt$ is the (signed) area under f between a and x. We assume $A(x+h)$ is defined for all h in some open interval containing 0. Assume f is continuous at x.

$$\begin{aligned} D_x A(x) &= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} \\ &= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} \end{aligned}$$

Now

$$h \cdot m_f \leq \int_x^{x+h} f(t)\,dt \leq h \cdot M_f$$

where m_f and M_f are the "minimum" and "maximum" of f over $[x, x+h]$. Since f is continuous at x, for h small enough m_f and M_f exist and $\to f(x)$ as $h \to 0$. Thus on dividing the inequality by h we see $\lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = f(x)$.