Gudermann function

A definition for the *Gudermann* function is:

$$\operatorname{gd} x = \int_0^x \operatorname{sech} t \, dt.$$

The function is also called the *Gudermannian* function or the *hyperbolic amplitude* function. Its domain is all x.

Using the substitution $u = \sinh t$, for which $du = \cosh t \, dt$, and $1 + u^2 = 1 + \sinh^2 t = \cosh^2 t$ we find an alternate definition for this function:

$$\begin{aligned}
\operatorname{gd} x &= \int_0^x \operatorname{sech} t \, dt = \int_0^x \frac{1}{\cosh t} \, dt = \int_{t=0}^x \frac{\cosh t}{\cosh^2 t} \, dt = \int_{u=0}^{\sinh x} \frac{du}{1+u^2} \\
&= \left(\arctan u\right)\Big|_{u=0}^{\sinh x} \\
&= \arctan \sinh x.
\end{aligned}$$

So

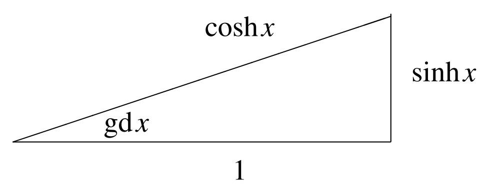
$$\operatorname{gd} x = \int_0^x \operatorname{sech} t \; dt = \arctan \sinh x. \tag{1}$$

The graph of gd is similar to the graph of arctan.

By the Fundamental Theorem of Calculus:

$$\frac{d}{dx} \operatorname{gd} x = \operatorname{sech} x. \tag{2}$$

Since $\operatorname{gd} x$ is an angle whose tangent is $\sinh x = (\sinh x)/1$ so we have a triangle as follows, where the hypotenuse of the triangle is $\sqrt{1 + \sinh^2 x} = \cosh x$:



From this triangle we read off the following relationships. They are significant since through the Gudermann (and without using complex numbers) they express hyperbolic functions in terms of trig functions.

$$\cos \operatorname{gd} x = \frac{1}{\cosh x} = \operatorname{sech} x \tag{3}$$

$$\sin \operatorname{gd} x = \frac{\sinh x}{\cosh x} = \tanh x \tag{4}$$

$$\tan \operatorname{gd} x = \sinh x \tag{5}$$

$$\sec \operatorname{gd} x = \cosh x \tag{6}$$

$$\csc \operatorname{gd} x = \frac{\cosh x}{\sinh x} = \coth x \tag{7}$$

$$\cot \operatorname{gd} x = \frac{1}{\sinh x} = \operatorname{csch} x \tag{8}$$

Here is yet another alternative expression for gd:

$$\operatorname{gd} x = 2\arctan e^x - \frac{\pi}{2}. \tag{9}$$

To prove this it suffices to show two things:

- Both sides are 0 at 0: $\operatorname{gd} 0 = \arctan\sinh 0 = \arctan 0 = 0$, while on the right side, $2\arctan e^0 - \frac{\pi}{2} = 2\arctan 1 - \frac{\pi}{2} = 2\frac{\pi}{4} - \frac{\pi}{2} = 0$.
- The derivatives of both sides are the same. The derivative of the right side is

  $$2 \cdot \frac{1}{1+(e^x)^2} \cdot e^x = \frac{2}{e^{-x}+e^x} = 1/\cosh x = \operatorname{sech} x,$$

  which happens to be the derivative of $\operatorname{gd} x$ by equation (2).

The inverse Gudermann function

The inverse function to $x = \operatorname{gd} y$ is $y = \operatorname{arcgd} x$, the *arc Gudermann* function. Its domain is $(-\frac{\pi}{2}, \frac{\pi}{2})$ and its graph is similar to the graph of tan on that interval. Its derivative, using equation (6), is

$$\begin{aligned}\frac{d\operatorname{arcgd} x}{dx} &= \frac{dy}{dx} = 1/\frac{dx}{dy} = 1/\frac{d}{dy}\operatorname{gd} y = 1/\operatorname{sech} y = \cosh y \\ &\overset{(6)}{=} \sec\operatorname{gd} y \\ &= \sec x. \end{aligned} \tag{10}$$

Integrating $\int_0^x$ and noting that $\operatorname{arcgd} 0 = 0$ (see (9)) gives:

$$\int_0^x \sec t\, dt = \operatorname{arcgd} x. \tag{11}$$

Next we determine several alternate expressions for arcgd, that is, for the integral of sec.

The six equations (3) to (8) give, respectively:

$$\operatorname{arcgd} x = \operatorname{arcsech}\cos x = \operatorname{arctanh}\sin x = \operatorname{arcsinh}\tan x \tag{12}$$
$$= \operatorname{arccosh}\sec x = \operatorname{arccoth}\csc x = \operatorname{arccsch}\cot x, \tag{13}$$

the proof of which we illustrate by using equation (4). The equation gives $\operatorname{arctanh}\sin\operatorname{gd} x = x$, from which, on replacing x with $\operatorname{arcgd} x$, we get $\operatorname{arctanh}\sin\operatorname{gd}\operatorname{arcgd} x = \operatorname{arcgd} x$, that is, $\operatorname{arctanh}\sin x = \operatorname{arcgd} x$. Note the pleasing symmetry:

$$\operatorname{gd} x = \arctan\sinh x \quad \text{and} \quad \operatorname{arcgd} x = \operatorname{arctanh}\sin x. \tag{14}$$

Similar formulas hold for the other five cases.

There are expression for arccosh and arctanh in terms of ln which give these formulas:

$$\operatorname{arcgd} x = \operatorname{arccosh} \sec x = \ln\left(\sec x + \sqrt{\sec^2 - 1}\right) = \ln(\sec x + \tan x), \tag{15}$$

$$\operatorname{arcgd} x = \operatorname{arctanh} \sin x = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right). \tag{16}$$

Equation (9) gives $\arctan e^x = \frac{1}{2}\operatorname{gd} x + \frac{\pi}{4}$, which rewrites as $x = \ln\tan\left(\frac{1}{2}\operatorname{gd} x + \frac{\pi}{4}\right)$. Replacing x with $\operatorname{arcgd} x$ and using $\operatorname{gd}\operatorname{arcgd} x = x$ gives:

$$\operatorname{arcgd} x = \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right). \tag{17}$$

In summary, from equations (11), (15), (16), (17):

$$\int_0^x \sec t \, dt = \operatorname{arcgd} x = \operatorname{arccosh} \sec x = \operatorname{arctanh} \sin x = \ln(\sec x + \tan x) = \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

These formulas for the integral of sec, especially the last one, have an interesting history, as they were discovered numerically in connection with map making by Mercator, Gunter, and Bond and proved by Newton's teacher Barrow and another famous mathematician of the time (Gregory).

Note that the antiderivative of sec is usually given with an absolute value:

$$\int \sec x \, dx = \ln|\sec x + \tan x| + c$$

but in our case the domain is $(-\pi/2, \pi/2)$ making the absolute value superfluous since there $\sec x + \tan x = \frac{1}{\cos x}(1 + \sin x)$ is a product of positives.