Householder Matrices

If $\mathbf{v}$ is a nonzero column vector, the square matrix

$$P = I - \frac{2}{\mathbf{v}^*\mathbf{v}}\mathbf{v}\mathbf{v}^*$$

is called a **Householder matrix**. Note that $\mathbf{v}^*\mathbf{v} = \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.
For example, if $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ then $P = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.

Householder matrices are connected to the Gram-Schmidt process. It is not hard to show that a Householder matrix is Hermitian and unitary.

A Householder matrix can be used to partially "zero out" a vector as we now show.
Let a nonzero vector $\mathbf{x} = \begin{pmatrix} a \\ b_2 \\ \vdots \end{pmatrix}$ be given.
Write $a = |a|\mu$ where complex number

$$\mu = \begin{cases} a/|a|, & \text{if } a \neq 0; \\ 1, & \text{if } a = 0. \end{cases}$$

Set

$$\mathbf{s} = \begin{pmatrix} |\mathbf{x}|\mu \\ 0 \\ \vdots \end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \mathbf{x} + \mathbf{s} = \begin{pmatrix} (|a| + |\mathbf{x}|)\mu \\ b_2 \\ \vdots \end{pmatrix}.$$

Clearly $\mathbf{v} \neq \mathbf{0}$, since $|\mathbf{x}| \neq 0$ and $\mu \neq 0$, so the Householder matrix P is defined.
Below we show $2\mathbf{v}^*\mathbf{x} = \mathbf{v}^*\mathbf{v}$, thus we have:

$$P\mathbf{x} = \mathbf{x} - 2\frac{\mathbf{v}^*\mathbf{x}}{\mathbf{v}^*\mathbf{v}}\mathbf{v} = \mathbf{x} - \mathbf{v} = -\mathbf{s}.$$

Thus P applied to vector $\mathbf{x}$ gives the "zeroed out" vector $-\mathbf{s}$.

To show $2\mathbf{v} \cdot \mathbf{x} = \mathbf{v} \cdot \mathbf{v}$, substitute $\mathbf{v} = \mathbf{x} + \mathbf{s}$ and expand. Thus show

$$2(\mathbf{x} \cdot \mathbf{x} + \mathbf{s} \cdot \mathbf{x}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{s} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{s} + \mathbf{s} \cdot \mathbf{s}.$$

But direct computation shows $\mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2 = |\mathbf{x}|^2|\mu|^2 = \mathbf{s} \cdot \mathbf{s}$ and $\mathbf{x} \cdot \mathbf{s} = \overline{a}|\mathbf{x}|\mu = |a||\mathbf{x}|$ which is real and so equals $\mathbf{s} \cdot \mathbf{x}$.