L'Hospital's Rule

Theorem 1 (L'Hospital's Rule) *If*

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$$

or

$$\lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x)$$

where $a = +\infty$ or $a = -\infty$ is allowed, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the latter limit exists or is $\pm\infty$.
[Implicitly we are assuming that near 'a' the expressions $f(x)/g(x)$ and $f'(x)/g'(x)$ are defined, that is, the derivatives exist and g and g' are non-zero.]

Proof. First we show that all versions can be reduced to the case $x \to 0^+$ with $f(x)$, $g(x) \to 0$ or $f(x)$, $g(x) \to \infty$ by making suitable substitutions u for x, among other things.

1. $x \to \infty$: Setting $u = 1/x$ gives $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{u\to 0^+} \frac{f(1/u)}{g(1/u)} \stackrel{L'H}{=} \lim_{u\to 0^+} \frac{f'(1/u)\cdot(-1)/u^2}{g(1/u)\cdot(-1)/u^2} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$.
2. $x \to -\infty$: As above but use $u = -1/x$.
3. $x \to 0$: Split as two separate limits $x \to 0^+$ and $x \to 0^-$. Handle the limit $x \to 0^-$ by substituting $u = -x$.
4. $x \to a$: Reduces to the previous by substituting $u = x - a$.
5. For the case of $f(x) \to -\infty$ or $g(x) \to -\infty$ replace f with $-f$ or g with $-g$.
6. For the case of the $\lim f(x)/g(x) = \pm\infty$, replace f by $-f$ or g by $-g$ to reduce this to $\lim f(x)/g(x) = \infty$. Then apply L'Hospital's rule to $\lim g(x)/f(x)$ instead.

Now we prove L'Hospital's Rule for $x \to 0+$, $f(x) \to 0$, $g(x) \to 0$ (or $f \to \infty$, $g \to \infty$). We need that $f'(x)$ and $g'(x)$ exist and $g(x)' \neq 0$ throughout some interval $(0, b]$ for some $b > 0$, so that $\frac{f'}{g'}$ makes sense.
In the case of $f(x) \to \infty$, $g(x) \to \infty$ define $F(x) = 1/f(x)$ and $G(x) = 1/g(x)$ and apply the following argument to F, G instead of f, g.
Under the implicit assumption, f and g are continuous on some interval $(0, b]$ and since $f(x) \to 0$, $g(x) \to 0$ we can extend/change the definition of f and g by $f(0) = 0 = g(0)$. Consider $H(x) = f(x)g(b) - g(x)f(b)$. Then $H(0) = f(0)g(b) - g(0)f(b) = 0$ and $H(b) = f(b)g(b) - g(b)f(b) = 0$. By Rolles' Theorem, for some c with $0 < c < b$, $H'(c) = 0$. $H'(x) = f'(x)g(b) - g'(x)f(b)$ which at $x = c$ becomes $0 = f'(c)g(b) - g'(c)f(b)$ so $f(b)/g(b) = f'(c)/g'(c)$. This result is known as Cauchy's Mean Value Theorem. Now let $b \to 0$. Then $c \to 0$ and relabelling b and c as x we get $\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)}$. $\square$

Remark on the proof: An easier argument is usually possible when $x \to 0^+$. Unfortunately this argument needs f', g' to be continuous at 0 and $g'(0) \neq 0$ which may not be the case in general:

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{(f(x) - f(0))/x}{(g(x) - g(0))/x} = \frac{f'(0)}{g'(0)} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)}.$$