Permutations with Restricted Positions and Rook Polynomials

A permutation of 1, 2, ..., n corresponds to a placing of n mutually noncapturing rooks on an n × n board, or equivalently, a matrix of 0’s and 1’s with exactly one 1 in each row and each column. For example, the left figure illustrates the permutation 3 2 4 1, where we put a rook R in the 1st row and 4th column because the 1 is in the 4th position of the permutation, etc.:

In a permutation, such as a derangement, we can indicate restricted, that is, forbidden positions by shading appropriate squares on the board. Any kind of permutation with restricted positions can be indicated this way. For example, the restricted positions D4 for a 4-derangement are illustrated on the right above.

Let C denote a board with shadings. In inclusion-exclusion, we would consider the set Ai, the set of all permutations in which rook i is in a forbidden position. As well we would consider multiple intersections of these kinds of sets, which leads us to consider the following.

Let r_k(C) be the number of arrangements of k rooks such that all k are in forbidden positions. If we let the remaining n−k rooks occupy any other mutually noncapturing positions (there are (n−k)! ways) then r_k(C)(n−k)! is precisely the S_k of the inclusion-exclusion formula. Thus by inclusion-exclusion:

The number of permutations with no objects in forbidden positions is

\[ r_0(C) n! - r_1(C)(n-1)! + r_2(C)(n-2)! + \cdots \]

The polynomial \( r(C) = r_0(C) + r_1(C)x + r_2(C)x^2 + \cdots + r_n(C)x^n \) is called the rook polynomial of C. Note \( r_0(C) = 1 \) always.

The main advantage of this approach is that \( r(C) \) can be computed relatively easily for a given pattern because there are a number of reductions available.

(I) If \( C = C_1 E_1 \)
\( E_2 \)
\( C_2 \)
where \( E_1 \) and \( E_2 \) are blank (no shaded squares) then \( r(C) = r(C_1) \cdot r(C_2) \).

This is essentially the multiplication rule as follows. Place i rooks in \( C_1 \) and \( k-i \) rooks in \( C_2 \). There are \( r_i(C_1) \cdot r_{k-i}(C_2) \) ways. Then add up cases — add over all the i’s.

(II) By a direct counting argument it can be shown that

\[ r(F_j) = 1 + \binom{j}{1} jx + \binom{j}{2} j(j-1)x^2 + \cdots + \binom{j}{j} j!x^j \]

where \( F_j \) is the size j board with all squares shaded. Namely, choose the rows for the rooks, and then j represents the columns available for the first rook, \( j-1 \) columns for second rook, etc.

(III) Let C be a board in which square q is shaded. Let \( C_q \) be the subboard obtained from C by removing the row and column of q and let \( C_e \) be C but with square q unshaded. Then

\[ r(C) = r(C_e) + r(C_q)x \]
since in the case where we do not put a rook on q we might as well remove q giving \( C_e \) and in the case where we do put a rook on q we have one fewer rook to give away — the reason for the x — and the remaining rooks must go on \( C_q \).
If $E$ is empty then $r(E) = 1$. Also $r(E) = 1 + x$. Applying (I) gives $r(D_j) = (1 + x)^j$ where $D_j$ is the $j$-derangement board, that is, the size $j$ board with just the main diagonal squares shaded.

**EXAMPLE:** Flip the 52 cards from a standard deck in succession while calling out the 13 ranks Ace, King, Queen, etc. in order, and restart at the Ace after you call out the rank two. What is the probability that you will never flip over a card with the called rank?

This is not a 52-derangement, although it is related. The board $C$ for this situation consists of a four by four array of $D_{13}$'s (see left figure). It can be rearranged into a more tractable board $C'$ (see right figure) as follows: Put all the Aces together, all the Kings together, etc. This is equivalent to calling out Ace four times, then King four times, etc. The board $C'$ is 13 copies of $F_4$ down the diagonal.

$C$  A K ··· A K ··· A K ··· A K ··· 
A K ··· A K ··· A K ··· A K ···  
A K ··· A K ··· A K ··· A K ···  
A K ··· A K ··· A K ··· A K ···  

$C'$  AAAAA KKKKK ···  
AAAA KKKK ···  
AAAA KKKK ···  
AAAA KKKK ···  
AAAA KKKK ···  

By (I) 13 times and (II) with $j = 4$ the rook polynomial is

$$(1 + 16x + 72x^2 + 96x^3 + 24x^4)^{13}$$

Expanding this out and replacing each $x^i$ by $(-1)^i(52 - i)!$ gives the number of arrangements that do not have forbidden positions. Using the computer algebra system MAPLE, this number was computed to be

1309302175551177162931045000259922525308763433362019257020678406144.

Dividing this number, $1.309 \times 10^{66}$, by $52!$ which is about $8.065 \times 10^{67}$ gives the probability to be approximately .0162327. In summary, you have about a one and a half percent chance of getting through the deck without a match.