Space Curve Example — a Helix

Name, General Formula, and Equivalents $(\equiv)$		**Value for right-handed Helix, radius** 3**, height** $2\pi 4$

$$\text{Position param. by time} \equiv \mathbf{r}(t) = (3\cos t, 3\sin t, 4t) \tag{1}$$

$$\text{Velocity} \equiv \mathbf{v} \equiv \dot{\mathbf{r}} = (-3\sin t, 3\cos t, 4) \tag{2}$$

$$\text{speed} \equiv \dot{s} = |\mathbf{v}| = |\dot{\mathbf{r}}| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} = 5 \tag{3}$$

$$\text{Tangent vector} \equiv \mathbf{T} \equiv \mathbf{r}' \equiv \dot{\mathbf{r}}/|\dot{\mathbf{r}}| = \left(\frac{-3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right) \tag{4}$$

$$\text{Acceleration} \equiv \mathbf{a} \equiv \ddot{\mathbf{r}} = (-3\cos t, -3\sin t, 0) \tag{5}$$

$$\text{arclength} \equiv s(t) \equiv \int_0^t |v|\,dt = \int_0^t 5\,dt = 5t \qquad \text{Usually very hard to find!} \tag{6}$$

$$\dot{s} = 5 \qquad \text{from previous but (3) more direct.} \tag{7}$$

$$\text{Position param. by arclength} \equiv \mathbf{r}(s) = \left(3\cos\frac{s}{5}, 3\sin\frac{s}{5}, \frac{s}{5}\right) \tag{8}$$

$$\textit{Key relationship:} \quad \mathbf{u}' = \dot{\mathbf{u}}/|\dot{\mathbf{r}}| = \frac{1}{5}\dot{\mathbf{u}} \quad \text{for every } \mathbf{u}, \text{ where } \mathbf{u}' = \frac{d\mathbf{u}}{ds}, \text{ and } \dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt}. \tag{9}$$

$$\text{First Frenet-Serret formula:} \quad \mathbf{T}' = \frac{1}{5}\dot{\mathbf{T}} = \frac{1}{5}\left(\frac{-3}{5}\cos t, \frac{-3}{5}\sin t, 0\right) \tag{10}$$

$$= \frac{3}{25}(-\cos t, -\sin t, 0) \equiv \kappa\mathbf{N} \tag{11}$$

$$\text{curvature} \equiv \kappa = |\mathbf{T}'| = \frac{3}{25} \qquad \text{from (11) and (14)} \tag{12}$$

$$\text{Normal vector} \equiv \mathbf{N} \equiv \mathbf{T}'/|\mathbf{T}'| = (-\cos t, -\sin t, 0) \qquad \text{from (11) since } |\mathbf{N}| = \tag{13}$$

$$\sqrt{\cos^2 t + \sin^2 t + 0^2} = 1; \text{ direction makes } \kappa \geq 0. \tag{14}$$

$$\text{Binormal vector} \equiv \mathbf{B} \equiv \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{-3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \left(\frac{4}{5}\sin t, \frac{-4}{5}\cos t, \frac{3}{5}\right) \tag{15}$$

$$\text{Third Frenet-Serret formula:} \quad \mathbf{B}' = \frac{1}{5}\dot{\mathbf{B}} = \frac{1}{5}\left(\frac{4}{5}\cos t, \frac{4}{5}\sin t, 0\right) \tag{16}$$

$$= \frac{-4}{25}(-\cos t, -\sin t, 0) = \frac{-4}{25}\mathbf{N} \equiv -\tau\mathbf{N} \tag{17}$$

$$\text{torsion} \equiv \tau = +\frac{4}{25} \qquad \text{from (17)} \tag{18}$$

$$\text{Second Frenet-Serret formula:} \quad \mathbf{N}' = \frac{1}{5}\dot{\mathbf{N}} = \frac{1}{5}(\sin t, -\cos t, 0) \tag{19}$$

$$= \frac{-3}{25}\left(\frac{-3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right) + \frac{4}{25}\left(\frac{4}{5}\sin t, \frac{-4}{5}\cos t, \frac{3}{5}\right) \tag{20}$$

$$= -\kappa\mathbf{T} + \tau\mathbf{B} \tag{21}$$

$$\text{Acc. components: } \mathbf{a} = \ddot{s}\,\mathbf{T} + (\dot{s})^2\kappa\mathbf{N} = 0\mathbf{T} + 5^2\frac{3}{25}\mathbf{N} = 0\mathbf{T} + 3\mathbf{N} \tag{22}$$

$$\textbf{Frenet-Serret formulas:} \quad \begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} -\frac{3}{25}\cos t & -\frac{3}{25}\sin t & 0 \\ \frac{1}{5}\sin t & -\frac{1}{5}\cos t & 0 \\ \frac{4}{25}\cos t & \frac{4}{25}\sin t & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{25} & 0 \\ -\frac{3}{25} & 0 & \frac{4}{25} \\ 0 & -\frac{4}{25} & 0 \end{pmatrix} \begin{pmatrix} -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \\ \frac{4}{5}\sin t & -\frac{4}{5}\cos t & \frac{3}{5} \end{pmatrix} \tag{24}$$

$$\text{Radius of curvature} \equiv 1/\kappa = \frac{25}{3} \tag{25}$$

$$\text{Radius of torsion} \equiv 1/\tau = \frac{25}{4} \tag{26}$$

$$\text{Total curvature} \equiv |\mathbf{N}'| = \sqrt{\kappa^2+\tau^2} = \sqrt{\left(\frac{3}{25}\right)^2 + \left(\frac{4}{25}\right)^2} = \sqrt{\frac{1}{25}} = \frac{1}{5} \tag{27}$$

$$\text{Osculating plane, } (\mathbf{x}-\mathbf{r})\cdot\mathbf{B} = 0: \quad \frac{4}{5}(\sin t)x - \frac{4}{5}(\cos t)y + \frac{3}{5}z = \frac{12}{5}t \tag{28}$$

$$\text{Normal plane, } (\mathbf{x}-\mathbf{r})\cdot\mathbf{T} = 0: \quad -\frac{3}{5}(\sin t)x + \frac{3}{5}(\cos t)y + \frac{4}{5}z = \frac{16}{5}t \tag{29}$$

$$\text{Rectifying plane, } (\mathbf{x}-\mathbf{r})\cdot\mathbf{N} = 0: \quad -(\cos t)x + (\sin t)y = -3 \tag{30}$$

Alternate formulas for κ, τ, T, N, B and components of acceleration

$$\mathbf{r} = (3\cos t, 3\sin t, 4t) \tag{31}$$

$$\dot{\mathbf{r}} = (-3\sin t, 3\cos t, 4) \tag{32}$$

$$\ddot{\mathbf{r}} = (-3\cos t, -3\sin t, 0) \tag{33}$$

$$\dddot{\mathbf{r}} = (3\sin t, -3\cos t, 0) \tag{34}$$

$$|\dot{\mathbf{r}}| = \sqrt{9^2+4^2} = 5 \tag{35}$$

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\sin t & 3\cos t & 4 \\ -3\cos t & -3\sin t & 0 \end{vmatrix} = (12\sin t, -12\cos t, 9) \tag{36}$$

$$|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}| = \sqrt{12^2+9^2} = 15 \tag{37}$$

$$(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\cdot\dddot{\mathbf{r}} = 36\sin^2 t + 36\cos^2 t + 0 = 36 \tag{38}$$

$$(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\times\dddot{\mathbf{r}} = (-75\cos t, -75\sin t, 0) \tag{39}$$

$$\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = 9\sin t\cos t - 9\cos t\sin t + 0 = 0 \tag{40}$$

$$\kappa = |\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|/|\dot{\mathbf{r}}|^3 = 15/5^3 = 3/25 \tag{41}$$

$$\tau = (\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\cdot\dddot{\mathbf{r}}/|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2 = 36/15^2 = 4/25 \tag{42}$$

$$\mathbf{a} = \ddot{\mathbf{r}} = \frac{\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|}\mathbf{T} + \frac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|}\mathbf{N} = \frac{0}{5}\mathbf{T} + \frac{15}{5}\mathbf{N} = 0\mathbf{T} + 3\mathbf{N} \tag{43}$$

$$\mathbf{T} = \dot{\mathbf{r}}/|\dot{\mathbf{r}}| = ((-3/5)\sin t, (3/5)\cos t, 4/5) \tag{44}$$

$$\mathbf{B} = \dot{\mathbf{r}}\times\ddot{\mathbf{r}}/|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}| = ((4/5)\sin t, -(4/5)\cos t, 3/5) \tag{45}$$

$$\mathbf{N} = (\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\times\dot{\mathbf{r}}/(|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}||\dot{\mathbf{r}}|) = (-\cos t, -\sin t, 0) \tag{46}$$