RESEARCH STATEMENT

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1. The Riemann Zeta function

My research activities fall in the domain of analytic number theory, in particular the distribution of prime numbers. Riemann wrote only one article on the theory of numbers, published in 1859, but it brought a simple and revolutionary approach to the distribution of the primes numbers. His fundamental idea was to extend Euler's formula to a complex variable. Thus, he sets :

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p} \left(1 - p^{-s}\right)^{-1}$$

for every complex s with $Re(s) > 1$, where p runs through all primes. This equality shows the intimate connection between the zeta-function and the prime numbers. The importance of this comes from the fact that the variable s is a complex number, so that $\zeta(s)$ may have an analytic continuation outside the region $\mathfrak{Re} s = \sigma > 1$, where the series converges. In fact, it is precisely information about $\zeta(s)$ when $\sigma \leq 1$ which is of crucial importance to understand the distribution of the primes.
Moroever, Riemann gave a meromorphic continuation of this function to all of $\mathbb{C}$ and he proved a functional equation relating $\zeta(s)$ to $\zeta(1-s)$. From this equation, he deduced that all complex (non-real) zeros of the zeta function lie in the strip $\{s \in \mathbb{C}, 0 \leq \mathfrak{Re} s \leq 1\}$ and that they are symmetrically distributed about the critical line $\mathfrak{Re} s = 1/2$ and the real axis.
The precise location of the zeros remains a mystery even today and represents one of the major problems in zeta-function theory. In fact, Riemann expressed the opinion that all the nontrivial zeros lie on the line $\mathfrak{Re} s = 1/2$. All attempts to prove this conjecture known as the Riemann Hypothesis (RH), have failed and it is not even known whether there is a σ_0 such that $1/2 \leq \sigma_0 < 1$ and $\zeta(s) \neq 0$ for $\mathfrak{Re} s > \sigma_0$.

This conjecture has inspired researchers to compute the first zeros of the zeta functions. Since calculations of Turing in 1956, on some of the earliest large scale computers, we know that all "low" zeros of the zeta function satisfy Riemann's hypothesis. The most recent calucations of S. Wedeniwski have demonstrated that the conjecture is true to height $\mathfrak{Im} s = 3.3 \cdot 10^9$ in the critical strip. However, to date the Riemann hypothesis remains neither proven nor disproven.

Although we cannot locate all the zeros, we can find a region containing no zeros of the Riemann zeta function. For the purpose of number theory, it is important to extend as far as possible this zero-free region of the zeta function. It appears that this problem is a question of extending as far as possible the sphere of influence of the Euler product beyond its actual area of convergence. In 1896, Hadamard and De La Vallée Poussin showed simultaneously but independently that the zeta-function never vanishes on the line $\mathfrak{Re} s = 1$. This assertion is the fundamental tool that permitted them to find an asymptotic for the number of primes less than x, namely $\pi(x)$:

$$\pi(x) \underset{x\to+\infty}{\sim} Li(x) \text{ with } Li(x) = \int_2^x \frac{dt}{\log t}.$$

This is called the Prime Number Theorem.

In 1899, De La Vallée Poussin enlarged this zero-free region to the left of the line $\mathfrak{Re} s = 1$, proving that there exists an absolute constant $R_0 > 0$ such that the Riemann zeta-function never vanishes in the region :

$$\mathfrak{Re} s \geq 1 - \frac{1}{R_0 \log\left(|\mathfrak{Im} s|\right)}, \quad |\mathfrak{Im} s| \geq 2.$$

He also established a numerical value for the constant : $R_0 = 34.82$. Reducing this value also improves the error term in the prime number theorem :

$$|\pi(x) - Li(x)| \ll x \exp\left(-\sqrt{\frac{\log x}{R_0}}\right).$$

In later works, the above zero-free region was enlarged by reducing the constant R_0. The most recent results were due to B.Rosser et L.Schoenfeld who established $R_0 = 9.646$ in 1975 (see [18]) and twenty-five years later, K. Ford (see [3], Theorem 4) obtained the value 8.463, by proving a good effective upper bound for ζ. All of these authors have followed the ideas of de la Vallée Poussin except for Ford. His work was based on Korobov and Vinogradov's upper bound for $\zeta(s)$ near to the line $\mathfrak{Re} s = 1$. Moreover, he proved a zero-free region of the form

$$\mathfrak{Re} s > 1 - \frac{1}{R_1 (\log|\mathfrak{Im} s|)^{2/3} (\log\log|\mathfrak{Im} s|)^{1/3}} \qquad (|\mathfrak{Im} t| \geq 10).$$

We notice that the classical zero-free region remains larger than the Korobov-Vinogradov's in the region $|\mathfrak{Im} s| \leq \exp(9412)$. This implies that the first inequality is especially useful for explicit results for the distribution of the primes, whereas the second inequality is better for asymptotic results. The focus of my research has been to enlarge the De La Vallée Poussin's zero-free region. I established:

Theorem 1 ([8]). *The Riemann zeta function never vanishes in the region:*

$$\mathfrak{Re} s \geq 1 - \frac{1}{R_0 \log(|t| + 2)} \qquad \text{where} \qquad R_0 = 5.697. \tag{1}$$

This result depends partially on numerical calculations. In effect, the optimal constant for this method is $R_0 = 5.653$.To better understand the methods presented here, I will review some of the past techniques applied in this domain.

The fundamental idea consists of comparing the order of magnitude of $\zeta(s)$ when s is close to its pole at $s = 1$ and also when s is close to the zero ρ_0 that we are trying to locate. This procedure is formalized by expressing $\zeta(s)$ as a function of its pole and its zeros. There are two methods here.

The first one consists of an equality relating sums over prime powers to sums over all non-trivial zeros of zeta. We usually refer to it as "global formula":

$$-\mathfrak{Re} \frac{\zeta'}{\zeta}(s) = \mathfrak{Re} \frac{1}{s-1} - \sum_{\zeta(\varrho)=0} \mathfrak{Re} \frac{1}{s-\varrho} + \frac{1+o(1)}{2} \log\left(|t| + 2\right), \tag{2}$$

where $Z(\zeta)$ is the set of non-trivial zeros of the zeta function. It can be obtained by Weierstrass-Hadamard type theorem. The important points are to control the size of the sum over the zeros and to reduce the numerical value of the coefficient factor in front of the log. From this point of view, Stechkin (cf. [19]) considers ζ at both the values s and $s+\delta$, where δ is a positive constant, and studies $\frac{\zeta'}{\zeta}(s)-\kappa\frac{\zeta'}{\zeta}(s+\delta)$. He is still able to control this new sum over the zeros under a suitable condition on κ (κ can not be larger than $1/\sqrt{5}$). The advantage of this method is that it multiplies the coefficient $1/2$ in (2), and therefore the final constant R_0, by $(1-\kappa)$. This is how Rosser & Schoenfeld obtained $R_0 = 9.646$ ([18]).

The second approach, due to Landau, is related to Jensen-type theorems. The sum over the zeros is restricted to those in a small circle centered close to $\mathfrak{Re}s = 1$. For this reason one usually refers to it as a "local formula". For $|t|$ sufficiently large:

$$-\mathfrak{Re}\frac{\zeta'}{\zeta}(s) \leq \mathfrak{Re}\frac{1}{s-1} - \sum_{\zeta(\varrho)=0, |s-\varrho|\leq\delta} \mathfrak{Re}\frac{1}{s-\varrho} + \phi\log\left(|t|+2\right), \tag{3}$$

where δ is a constant depending on ϕ and t. In order to this result, one needs to bound zeta on the half line. In this way, K. Ford proves $R_0 = 8.463$ ([3]). He also introduces a weight f in the sum of $-\frac{\zeta'}{\zeta}(s) = \sum_{n\geq 1}\Lambda(n)n^{-s}$ and hence studies $\sum_{n\geq1}\Lambda(n)f(\log n)n^{-s}$ instead. This last sum will now be related to a sum over zeros of the zeta function evaluated at $F(s-\varrho)$ where F is the Laplace transform of f. The expression $F(s-\varrho)$ will now play the role of $\frac{1}{s-\varrho}$ in (3) and the radius δ only depends of the choice of F and ϕ. He employs a weight function f, previously used by Heath-Brown in [6].

My approach to this problem is hybrid in the sense that I establish a smoothed version of Stechkin, establishing an explicit formula for: $\sum_{n\geq1}\Lambda(n)f(\log n)n^{-s}(1-n^{-\delta})$. I use Heath-Brown's weight but without any condition on the sum over the zeros, turning Ford's formula into a global one. The key point is then to control its size. This is made possible by a generalisation of Stechkin's idea. This allows one to reduce the log contribution and therefore the final constant R_0.

2. Dirichlet L functions

One of the principal interests of the method presented in the last section, contrary to that of Kevin Ford, is that it generalizes to Dirichlet L-functions. These functions play an important role in the study of primes in arithmetic progressions. In fact, Dirichlet exploited them to prove that there are infinitely many prime numbers in an arithmetic progression $\{a+nq\}$. An interesting question is the location of the zeros of the Dirichlet L-functions. The conjecture that all the zeros of Dirichlet L-functions satisfy $\mathfrak{Re}s = 1/2$ is called the Generalized Riemann Hypothesis. Several researchers have numerically verified it for $|\mathfrak{Im}s| \leq T_q$. However, in this case the calculations are rended complicated due to the dependence on q. One of the most recent calculations verifying this hypothesis is due to M. Bennett who was motivated by applications in diophantine approximation. Like the zeta function, Dirichlet L-functions satisfy an Euler product and a functional equation. Using the same method than for zeta, one can establish a zero-free region of the form

$$\mathfrak{Re}s \geq 1 - \frac{1}{R_1\log(q\max(1,|\mathfrak{Im}s|))}, \tag{4}$$

except for at most one real zero that corresponds to a real Dirichlet character. Unfortunately, the standard proof fails for real zeros of real characters. To date, this problem has remained unresolved. This possible zero, also called Siegel zero, is of central importance in number theory since it would be the worst counterexample to the Riemann Hypothesis. It became one of the most challenging problem to figure out if it exists or not. It is widely expected that it does not. Many investigations have been done under the assumption of its existence. The results obtained are surprisingly better than under the Riemann Hypothesis condition. This lead to think that "it is too good to be true" and that maybe one of the result would be disproved, implying the non-existence of this zero.

One of the reason for proving zero free regions is to apply them to arithmetic applications. For example, a famous result of Linnik asserts that, for q sufficiently large and a coprime to q, the least prime $P(a,q)$ in the arithmetic progression a mod q, satisfies

$$P(a,q) \ll q^C$$

for some explicit constant C. The first value given for C was 10 000 in 1957 and over the years there have been many improvements of this result. The current record, due to Heath-Brown, is $C = 5.5$. To prove this, he established that the function $\mathcal{L}_q(s) = \prod_{\chi mod\ q} L(s,\chi)$ possesses at most one zero in the region described in (4) for $R_1 = 2.873$ and q asymptotically large. One of the key points of his proof is to bound $L(s,\chi)$ for s near the 1-line. For this, he improves on Burgess' bounds for character sums and this forces q to be asymptotically large.

Here, we shall employ another strategy since we are aiming to obtain a result valid for all q. In 2002, I established the following result:

Theorem 2 ([7]). *For each integer $q \ge 2$, the function $\mathcal{L}_q(s)$ possesses at most one zero in the region:*

$$\mathfrak{Re} s \ge 1 - \frac{1}{R_1 \log\big(q \max(1, |\mathfrak{Im} s|)\big)}, \quad \textit{where} \quad R_1 = 6.436, \tag{5}$$

improving on McCurley's result: $R_1 = 9.646$ ([11]). Here we apply the same arguments that we applied to the zeta function. However, some complications arise in the case when the imaginary parts of the zeros of the L-function are small. In this case, a different argument must be used. Since then, I improved R_1 to 6.396 (in [9]). I also gave an explicit version of the Deuring-Heilbronn repulsion phenomenon. It shows that if a Siegel zero exists, then it repulses the other close by zeros further to the left in the critical strip. We have the following result:

Theorem 3 ([9]). *Let χ_1 and χ_2 be distinct real primitive characters with conductors q_1 and q_2. If β_1 and β_2 are real zeros corresponding to $L(s,\chi_1)$ and $L(s,\chi_2)$ then*

$$\min(\beta_1,\beta_2) \le 1 - \frac{1}{R' \log(q_1 q_2)} \qquad \textit{where} \qquad R' = 2.046. \tag{6}$$

In this last theorem, we deduce immediately that for a given modulus there is at most one exceptional zero. Under certain conditions there is a most one exceptional zero given any two moduli. More precisely, if q_1 and q_2 correspond to exceptional characters then $q_2 > q_1^{2.16}$.

This phenomenon occurs also for non-real characters, but in the case of "small" imaginary parts:

Theorem 4 ([7])**.** *If $q \geq 10^6$ is a non-exceptional modulus, $n \leq 4$, χ a non-principal primitive character of conductor q and if $\varrho_k = \beta_k + i\gamma_k$, $1 \leq k \leq n$, are n zeros of $L(s,\chi)$ such that $1/2 < \beta_n < \beta_{n-1} < ... < \beta_1$ and $|\gamma_k| \leq \frac{1}{\log q}$ then β_n satisfy*

$$\beta_n \leq 1 - \frac{1}{R_n \log q} \tag{7}$$

where the values of R_n are given in the following table:

n	2	3	4
R_n	3.972	1.797	1.245

This says that there are at most n zeros in the region

$$\mathfrak{Res} \leq 1 - \frac{1}{R_n \log q} \quad \text{and} \quad |\mathfrak{Ims}| \leq \frac{1}{\log q}. \tag{8}$$

3. SHORT INTERVALS CONTAINING PRIMES IN ARITHMETIC PROGRESSIONS AND AN APPLICATION

The explicit nature of the above theorems allows one to deduce information on the primes in an intermediate range which lies beyond the range of numerical verifications by computer calculation. Note that known asymptotics results for primes only remain true in a much larger range.

Riemann's explicit formula relates primes in arithmetic progression and zeros of Dirichlet L-functions. In the particular case of the arithmetic progression $\{a + nq\}$ we define

$$\psi(x;q,a) = \sum_{\substack{p^k \leq x \\ p^k \equiv a \, (mod \, q)}} \log p \ .$$

This function is closely related to the associated prime counting function, however it is technically easier to study. One way to establish that the interval $[e^x, e^{x+\epsilon}]$ contains a prime $p \equiv a \pmod q$ would be to determine a condition on x such that

$$\psi\left(e^{x+\epsilon}; q, a\right) - \psi\left(e^x; q, a\right) = \sum_{\substack{e^x < n \leq e^{x+\epsilon} \\ n \equiv a[q]}} \Lambda(n) \tag{9}$$

is larger than a small positive constant. Riemann showed how to express this function as a sum of zeros of the Dirichlet L-function by using complex analytic methods. We have

$$\begin{aligned}\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi \text{ mod } q} \overline{\chi}(a) \sum_{|\gamma|<T} \frac{x^\varrho}{\varrho} \\ + \mathcal{O}\left(\frac{x}{\phi(q)T} \log^2(qx) + \frac{1}{\phi(q)} x e^{-c_1\sqrt{\log x}}\right).\end{aligned} \tag{10}$$

The previous method, initiated by Rosser, was approximating ψ via succesive integral averaging. In fact, this method amounts to weighting the primes with a smooth function. Our approach will be to introduce directly a smooth positive weight into the difference $\psi\left(e^{x+\epsilon}; q, a\right) - \psi\left(e^x; q, a\right)$:

$$\sum_{\substack{e^x < p^k \leq e^{x+\epsilon} \\ p^k \equiv a[q]}} \frac{\log p}{p^k} f(k \log p). \tag{11}$$

We choose the function f so that it has compact support contained in $[x, x+\epsilon]$ and so that the peak of the function is near the prime we want to locate. We have an explicit formula for the sum (11)

$$\frac{F(0)}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \sum_{\varrho \in Z(\chi)} F(1-\varrho) + o(1/\phi(q)), \tag{12}$$

where F is the Laplace transform of f and $Z(\chi)$ the set of non-trivial zeros of $L(s,\chi)$. We shall bound carefully each of the above term and obtained that it is positive when

$$x \geq \alpha \log^2 q \text{ and } \alpha \geq R\left(1 + \frac{\log r(\epsilon, q)}{\log q}\right),$$

where $r(\epsilon, q) = o(q)$. We have the following result:

Theorem 5 ([10]). *Let $q \geq 2$ be a non-exceptional modulus and let $(a,q) = 1$. There exists $\alpha > 0$ and $\epsilon > 0$, such that, if $x \geq \alpha \log^2 q$, then the interval $[e^x, e^{x+\epsilon}]$ contains a prime $p \equiv a \pmod q$. The positive constants α and ϵ depend only on q.*

We list their values in Table 1 of [10]. For example, when $q = 10^{25}$, McCurley had earlier obtained $\alpha = 10.88$ (see Table 1 of [11]). With our new smoothing function and his zero-free region ($R = 9.65$), his result may be improved to $\alpha = 9.88$. With our new zero-free region we reduce this to $\alpha = 6.555$.

Note that an explicit bound for the size of the least prime $p \equiv a \pmod q$, namely $P(a,q)$, follows immediately:

$$P(a,q) \leq e^{\alpha \log^2 q}. \tag{13}$$

There exists a stronger result than (13) and we refer the reader to the work of Heath-Brown on the subject. In [6], he proved:

$$P(a,q) \ll q^{5.5}.$$

However, (13) is completely explicit and therefore it can be applied to solve some effective problems.

We are now interested to Waring's problem for sums of seven cubes. In 1943, Linnik proved in [13] that every sufficiently large integer may be represented as a sum of seven cubes. It is widely expected that it is also true for any integer. In 1951, Watson simplified Linnik's proof by using a lemma establishing some conditions on n to be represented as sum of seven cubes. These conditions consist in finding prime integers in arithmetic progression as small as possible. For example, McCurley [12] found $n \geq \exp(1\,077\,334)$ and Ramaré [16] $n \geq \exp(205\,000)$. These authors use Chebyshev's estimates for $\theta(x;q,a)$. We replace this argument with our result concerning small intervals containing a prime mod q. And since this assertion is only proven for non-exceptional modulus, we give an explicit description of the scarcity of exceptional moduli. We prove:

Theorem 6 ([10]). *Every integer integer n larger than $N_0 = \exp(111\,610)$ is a sum of seven cubes.*

4. Research plan

4.1. **Vinogradov's bound for the three primes Goldbach's conjecture.** The three primes conjecture, posed in 1742 in a letter from Goldbach to Euler, states that every odd integer ≥ 9 can be written as a sum of three odd primes. Assuming the Generalized Riemann Hypothesis, Hardy and Litttlewood proved in 1922 that it was true for all sufficiently large odd integers. In 1937, Vinogradov successfully removed the GRH and showed that it was true for any integer $n \geq V$, where $V = 10^{7\cdot 10^6}$. This numerical value is far from satisfactory and we would like to lower the value for V considerably until it falls within the range of computability of the current fastest computers. The most recent numerical investigation by Y. Saouter in 1998 verifies the Goldbach conjecture for all odd integers $\leq 10^{20}$. Since this is almost a decade old it is likely that this value can be improved. Currently the best value obtained for V is

$$V = 10^{1350}$$

which was obtained by Liu and Wang in 2002 in [15]. The proof is based on the Hardy-Littlewood circle method. The idea is to study a weighted version of the number of representation of n as a sum of three primes: $n = p_1 + p_2 + p_3$, namely:

$$r(n) = \sum_{n=p_1+p_2+p_3} (\log p_1)(\log p_2)(\log p_3).$$

We can express $r(n)$ as a Fourier coefficient of the function:

$$S(\alpha) = \sum_{p\leq n} (\log p) e^{2i\pi p\alpha}.$$

The circle method consists in studying the expression $e^{2i\pi\alpha}$ on the circle. When α is close to rational numbers $\frac{a}{q}$ with small denominator we expect to obtain the principal contribution. This estimate is directly related to a completely explicit result on the distribution of primes in arithmetic progressions and therefore on the zeros of Dirichlet L-functions. Note that the value $V = 10^{1350}$ was obtained by applying the older zero-free region of McCurley [11]. My improvement of McCurley's result in Theorem 2 already gives

$$V = 10^{803}.$$

However, there is hope of improving this even more. For example, we could work with a weighted version of $S(\alpha)$. The weight would be chosen to smooth the primes and thus improve the analysis of the error terms.

4.2. **Distribution of zeros of Dirichlet L-functions.** The complete resolution of Goldbach's problem requires further information on the distribution of the zeros of Dirichlet L-functions. Liu and Wang proved several theorems concerning the distribution of these zeros in [14]. They worked on this subject in order to prove background material required for their article on Vingradov's bound. Many of their theorems can be improved thanks to the approach I developed in proving Theorems 1 and 2. Their work is concerned with the fact that the zeros do not cluster near the edge of the zero-free region.

One of their results asserts that the region

$$\mathfrak{Re} s \geq 1 - \frac{1}{r\log q},\ |\mathfrak{Im} s| \leq 1,$$

contains a small number of zeros with various explicit values of r. For example, when $r = 3.85$ there are at most 4 zeros in the region. However, it is clear that the methods from my thesis will improve this result.

Another of their results consists in bounding the zero-density function $N(\alpha, q, y)$ which counts the zeros ϱ of the L-functions of modulus q in the rectangle:

$$\alpha \leq \mathfrak{Re}\varrho \leq 1, \ |\mathfrak{Im}\varrho| \leq y.$$

Two methods are used here, according to whether α is close to 1 or not. In particular, when α is close to one we can study this using our methods from Theorems 1 and 2. When α is further from one more classical methods can be applied. However, I believe I can can improve the current best results for $N(\alpha, q, y)$.

4.3. **Selberg's sieve method and a zero density estimate.** The density I want to study counts the number $N(\lambda)$ of characters associated to a fixed modulus q that possess a zero in the region $1 - \lambda/\log q < \mathfrak{Re}s < 1$ and $|\mathfrak{Im}s| \leq 1$. Earlier results are due to Graham [5] and Heath-Brown [6] and they are of the type

$$N(\lambda) \leq \frac{c_1}{\lambda} e^{c_2\lambda},$$

where c_1 and c_2 are numerically computable contants and $\lambda \leq \log\log\log q$, for q sufficiently large. This density appears in their article concerning Linnik's constant ([5] and [6]). Therefore, any improvements on one of these constants would lead to an improvement of the Linnik constant. One of the key points in estimating $N(\lambda)$ is to establish a Selberg type identity, that is to say to find an appropriate approximation of the Möbius function in our context. In fact, Heath-Brown gives several hints at the end of his article on how to improve the estimate of $N(\lambda)$.

4.4. **Chebotarev's density.** One of my immediate projects is to generalize the above method of obtaining zero-free regions for Hecke L-functions. These are L-functions that arise in algebraic number theory and encode information concerning number fields. It would be reasonable to try our method on these L-functions since they possess an Euler Product, a functional equation. They also satisfy an explicit formula that relates them to all their zeros: it has been established by A.Weil in [20]. Let L/K be an abelian Galois extension and let $G = Gal(L/K)$. Let C denote a conjugacy class in G. For each prime ideal $\mathfrak{p}$ in K there is a corresponding Frobenius symbol $\sigma_{\mathfrak{p}}$. Chebotarev's density theorem tells us that the proportion of primes $\mathfrak{p}$ that satisfy $\sigma_{\mathfrak{p}} = C$ is $|C|/|G|$. Montgomery, Lagarias, and Odlyzko proved that the least prime $\mathfrak{p}$ with $\sigma_{\mathfrak{p}} = C$ satisfies

$$N_{K/\mathbb{Q}}(\mathfrak{p}) \leq 2d_L^c$$

where d_L is the field discriminant and c is an effective constant. I am interested in giving an explicit value for c or even trying to improve the above estimate. Some of the tools involved in the above argument are the Deuring-Heilbronn phenomenon concerning repulsion of zeros and also effective zero-free regions of Hecke L-functions.

4.5. **Horizontal distribution of zeros of the Riemann Zeta function.** Another of my immediate projects depends on a recent article of Granville and Soundararajan [4]. They investigate the problem of character sums and of the Polya-Vinogradov inequality:

$$M(\chi) := \max_{x} \left| \sum_{n\le x} \chi(n) \right| \ll \sqrt{q}\log q,$$

where χ is a primitive character modulo q. Surprisingly, they improve this bound for odd characters. And they highlight a very interesting new phenomenon: they prove how the size of $M(\chi)$ among a family of characters χ is related to the structure of this family. To prove this, they construct an object measuring the distance between the characters. This distance function also satisfies a triangle inequality from which we can derive a new trigonometric inequality which relates zeros of Zeta lying on a common vertical line close to $\mathfrak{Res} = 1$ and at different heights. Applying their ideas, I hope to establish new results concerning the repulsion phenomenon for the zeros of the Riemann zeta function.

4.6. **Chebyshev's bounds.** Chebyshev's bounds are explicit estimates on the counting prime sums θ and ψ, of the form:

$$\max_{1\le y\le x_1} |\psi(y)-y| < \epsilon_1 x_1, \quad \max_{1\le y\le x_2} \left|\psi(y;q,a) - \frac{y}{\phi(q)}\right| < \epsilon_2 \frac{x_2}{\phi(q)}, \tag{14}$$

where $0 < \epsilon \le 1$ and $x \ge x_0$ are computable. Dusart, in [1] and [2], made the last investigations for the primes and Ramaré and Rumely, in [17], for primes in arithmetic progressions. These types of results have many interesting applications and it would be of use to update these results. For example, Ramaré obtained a very good effective version of S'nirel'man's theorem. This problem is related to the Goldbach conjecture: S'nirel'man proved that there exists a constant C such that every integer larger than 1 is a sum of at most C prime numbers. Ramaré's effective version of this theorem was established with $C = 7$. This project would be appropriate for a strong master student.

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