EXPLICIT ZERO DENSITY FOR THE RIEMANN ZETA FUNCTION

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Abstract. Let $N(\sigma,T)$ denote the number of nontrivial zeros of the Riemann zeta function with real part greater than σ and imaginary part between 0 and T. We provide explicit upper bounds for $N(\sigma,T)$ commonly referred to as a zero density result. In 1937, Ingham showed the following asymptotic result $N(\sigma,T)=\mathcal{O}(T^{\frac{8}{3}(1-\sigma)}(\log T)^5)$. Ramaré recently proved an explicit version of this estimate. We discuss a generalization of the method used in these two results which yields an explicit bound of a similar shape while also improving the constants.

1. Introduction

Throughout this article $\zeta(s)$ denotes the Riemann zeta function and ϱ denotes a non-trivial zero of $\zeta(s)$ lying in the critical strip, $0<\mathfrak{Re}(s)<1$. Let $\frac{1}{2}<\sigma<1, T>0$, and define

$$N(\sigma,T)=\#\{\varrho=\beta+i\gamma:\ \zeta(\varrho)=0, 0<\gamma<T \text{ and } \sigma<\beta<1\}. \tag{1.1}$$

We shall prove a non-trivial, explicit upper bound for $N(\sigma,T)$. Such a bound is commonly referred to as a zero-density estimate. We denote RH the Riemann Hypothesis and RH(H_0) the statement:

$$\text{RH}(H_0):\ \text{all non-trivial zeros } \varrho \text{ of } \zeta(s) \text{ with } |\mathfrak{Im}(\varrho)|\le H_0 \text{ satisfy } \mathfrak{Re}(\varrho)=\frac{1}{2}. \tag{1.2}$$

Currently, the best published value of H_0 for which (1.2) is true is due to David Platt [19]:

$$H_0=3.0610046\cdot 10^{10}$$

with $N(H_0)=103\,800\,788\,359$. Other strong evidence towards the RH is the large body of zero-density estimates for $\zeta(s)$. Namely, very good bounds for $N(\sigma,T)$ in various ranges of σ.

Let $\sigma>\frac{1}{2}$. In 1913 Bohr and Landau [2] showed that

$$N(\sigma,T)=\mathcal{O}\left(\frac{T}{\sigma-\frac{1}{2}}\right) \tag{1.3}$$

for T asymptotically large. This result implies that for any fixed $\varepsilon>0$, almost all zeros of $\zeta(s)$ lie in the band $|\frac{1}{2}-\mathfrak{Re}(s)|<\varepsilon$. This was improved in 1937 by Ingham [12], who showed

$$N(\sigma,T)=\mathcal{O}\left(T^{(2+4c)(1-\sigma)}(\log T)^5\right) \tag{1.4}$$

assuming that $\zeta(\frac{1}{2}+it)=\mathcal{O}\left(t^{c+\epsilon}\right)$. In particular, the Lindelöf Hypothesis $\zeta(\frac{1}{2}+it)=\mathcal{O}\left(t^{\epsilon}\right)$ implies that $N(\sigma,T)=\mathcal{O}\left(T^{2(1-\sigma)+\epsilon}\right)$, also known as the Density Hypothesis. There is a prolific

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literature on the bounds for $\zeta(s)$, starting with the convexity bound of $c = \frac{1}{4} = 0.25$ (Lindelöf), the first subconvexity bound of Hardy & Littlewood [8] $c = \frac{1}{6} = 0.1666\ldots$, to some more recent results of Huxley [10] (2005) $c = \frac{32}{205} = 0.1560\ldots$ and of Bourgain [3] (2017) $c = \frac{13}{84} = 0.1547\ldots$. In addition, there are also many articles on estimates for $N(\sigma, T)$. A selection of some notable results may be found in [10], [11], [13], and [3]. On the other hand, there are few explicit bounds for $N(\sigma, T)$. We refer the reader to a result of the first author [14] for an explicit version of Bohr and Landau's bound. The method provides two kind of results: for T asymptotically large, as in $N(0.90, T) \le 0.4421T + 0.6443 \log T - 363\,301$, and for T taking a specific value, as in $N(0.90, H_0) < 96.20$. These bounds are useful to improve estimates of prime counting functions, as in [5], [4], [20], [26] and in [15] to find primes in short intervals. Ramaré had earlier proven a version of (1.4) in his D.E.A. memoire, which remained unpublished until recently. Let $\sigma \ge 0.52$ be fixed. In [24] he proves [1] that for any $T \ge 2000$

$$N(\sigma, T) \le 965(3T)^{\frac{8(1-\sigma)}{3}} (\log T)^{5-2\sigma} + 51.5(\log T)^2, \tag{1.5}$$

which gives $N(0.90, T) < 1293.48(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 51.50(\log T)^2$, which gives the bound for $T = H_0$: $N(0.90, H_0) < 2.1529 \cdot 10^{10}$. The purpose of this article is to bound $N(\sigma, T)$ by applying Ingham's argument with a general weight and to improve both [14] and [24].

Theorem 1.1. *Let* $\frac{10^9}{H_0} \le k \le 1, d > 0, H \in [1002, H_0)$, $\alpha > 0$, $\delta \ge 1$, $\eta_0 = 0.23622\ldots$, $1 + \eta_0 \le \mu \le 1 + \eta$, *and* $\eta \in (\eta_0, \frac{1}{2})$ *be fixed. Let* $\sigma > \frac{1}{2} + \frac{d}{\log H_0}$.
Then there exist $\mathcal{C}_1, \mathcal{C}_2 > 0$ *such that, for any* $T \ge H_0$,

$$N(\sigma, T) \le \frac{(T-H)(\log T)}{2\pi d} \log\left(1 + \frac{\mathcal{C}_1 (\log(kT))^{2\sigma} (\log T)^{4(1-\sigma)} T^{\frac{8}{3}(1-\sigma)}}{T-H}\right) + \frac{\mathcal{C}_2}{2\pi d} (\log T)^2, \tag{1.6}$$

where $\mathcal{C}_1 = \mathcal{C}_1(\alpha, d, \delta, k, H, \sigma)$ *and* $\mathcal{C}_2 = \mathcal{C}_2(d, \eta, k, H, \mu, \sigma)$ *are defined in* (4.72) *and* (4.73). *Since* $\log(1 + x) \le x$ *for* $x \ge 0$, (1.6) *implies*

$$N(\sigma, T) \le \frac{\mathcal{C}_1}{2\pi d} (\log(kT))^{2\sigma} (\log T)^{5-4\sigma} T^{\frac{8}{3}(1-\sigma)} + \frac{\mathcal{C}_2}{2\pi d} (\log T)^2. \tag{1.7}$$

In addition, numerical results are displayed in tables in Section 5.

For instance (1.7) gives $N(0.90, T) < 11.499(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 3.186(\log T)^2$, and (1.6) gives $N(0.90, H_0) < 130.07$. This improves previous results both numerically and methodologically (one of the key ingredients is the choice of a more efficient weight function in Ingham's method). Note that choosing $k < 1$ and optimizing in H can provide extra improvements to (1.5). In addition, we prove a stronger bound for the argument of a holomorphic function. We now explain the main ideas to prove Theorem 1.1.

[1] Equation (1.1) [24, p. 326] gives the bound $N(\sigma, T) \le 4.9(3T)^{\frac{8(1-\sigma)}{3}} (\log T)^{5-2\sigma} + 51.5(\log T)^2$. However, there is a mistake in [24]. The authors have been in communication with Professor Ramaré and he has sent us a proof of the revised inequality (1.5).

2. Setting up the proof

2.1. **Littlewood's classical method to count the zeros.** Let $h(s) = \zeta(s)M(s)$ where $M(s)$ is entire and

$$N_h(\sigma, T) = \#\Big\{\varrho' = \beta' + i\gamma' \in \mathbb{C} \;:\; h(\varrho') = 0, \sigma < \beta' < 1, \text{ and } 0 < \gamma' < T\Big\}. \tag{2.1}$$

Then for a parameter $H \in (0, H_0)$, we have by (1.2) that

$$N(\sigma, T) = N(\sigma, T) - N(\sigma, H) \le N_h(\sigma, T) - N_h(\sigma, H)$$

for $T \ge H_0$. We compare the above number of zeros for h to its average:

$$N_h(\sigma, T) - N_h(\sigma, H) \le \frac{1}{\sigma - \sigma'} \int_{\sigma'}^{\mu} (N_h(\tau, T) - N_h(\tau, H))\, d\tau$$

where $\mu > 1$ and σ' is a parameter satisfying $\frac{1}{2} < \sigma' < \sigma$. Let $\mathcal{R}$ be the rectangle with vertices $\sigma' + iH$, $\mu + iH$, $\mu + iT$, and $\sigma' + iT$. We apply the classical lemma of Littlewood as stated in [25, (9.9.1)]:

$$\int_{\sigma'}^{\mu} \Big(N_h(\tau, T) - N_h(\tau, H)\Big) d\tau = -\frac{1}{2\pi i} \int_{\mathcal{R}} \log h(s) ds. \tag{2.2}$$

Thus

$$N(\sigma, T) \le \frac{1}{2\pi(\sigma - \sigma')} \Big(\int_H^T \log|h(\sigma' + it)| dt + \int_{\sigma'}^{\mu} \arg h(\tau + iT) d\tau - \int_{\sigma'}^{\mu} \arg h(\tau + iH) d\tau - \int_H^T \log|h(\mu + it)| dt \Big). \tag{2.3}$$

As T grows larger, the main contribution arises from the first integral. The second and third integrals can be treated by using a general result for bounding $\arg f(s)$ for f a holomorphic function. To do this we give an improvement of a lemma of Titchmarsh [25, p. 213] (see Proposition 4.10 and Corollary 4.11 below). The fourth integral can be estimated with a standard mean value theorem for Dirichlet polynomials (see Lemma 3.6). A key goal is to minimize the above expression over admissible functions h. We now give an idea of how to estimate the first integral in (2.3).

2.2. **How the second mollified moment of $\zeta(s)$ occurs.** Let $X \ge 1$ be a parameter and define the mollifier to be

$$M_X(s) = \sum_{n \le X} \frac{\mu(n)}{n^s} \tag{2.4}$$

where $\mu(n)$ is the Möbius function. Note that this is a truncation of the Dirichlet series for $\zeta(s)^{-1}$. These mollifiers were invented by Bohr and Landau [2] to help control the size of $\zeta(s)$ in the critical strip. Futhermore, let

$$f_X(s) = \zeta(s)M_X(s) - 1. \tag{2.5}$$

Note that the series expansion for f_X is given by

$$f_X(s) = \sum_{n>X} \Big(\sum_{\substack{d|n \\ d\le X}} \mu(d) \Big) n^{-s} = \sum_{n\ge 1} \frac{\lambda_X(n)}{n^s}, \tag{2.6}$$

$$\text{with} \quad \lambda_X(n) = 0 \ \text{ if } n \le X, \quad \lambda_X(n) = \sum_{\substack{d|n \\ d\le X}} \mu(d) \ \text{ if } n > X. \tag{2.7}$$

We shall choose $h = h_X$ with

$$h_X(s) = 1 - f_X(s)^2 = \zeta(s)M_X(s)(2 - \zeta(s)M_X(s)). \tag{2.8}$$

Since we have

$$\frac{1}{b-a}\int_a^b \log f(t)dt \le \log\left(\frac{1}{b-a}\int_a^b f(t)dt \right),$$

for any f non-negative and continuous, and $|h_X(s)| \le 1 + |f_X(s)|^2$, we deduce that

$$\int_H^T \log\big(|h_X(\sigma' + it)|\big)\, dt \le (T-H)\log\left(1 + \frac{1}{T-H}\int_H^T |f_X(\sigma' + it)|^2 dt \right). \tag{2.9}$$

We denote

$$F_X(\sigma, T) = \int_0^T |f_X(\sigma + it)|^2 dt \text{ where } \sigma \ge \frac{1}{2}. \tag{2.10}$$

To resume, the key point for getting a good bound on $N(\sigma,T) - N(\sigma,H)$ is to obtain a good bound for $F_X(\sigma,T)$. Following a classical method due to Ingham we compare it to a smoothed version of itself.

2.3. **Ingham's smoothing method.** Let σ_1 and σ_2 be such that $\sigma_1 < \sigma < \sigma_2$. Let $T > 0$ and $g = g_T$ be a non-negative, real valued function, depending on the parameter T, and holomorphic in $\sigma_1 \le \mathfrak{Re}(s) \le \sigma_2$. We define

$$\mathcal{M}_{g,T}(X,\sigma) = \int_{-\infty}^{+\infty} |g(\sigma + it)|^2 |f_X(\sigma + it)|^2 dt. \tag{2.11}$$

We shall consider g of a special shape. For $\alpha, \beta > 0$, assume that there exist positive functions ω_1, ω_2 such that g satisfies, for all $\sigma \in [\sigma_1, \sigma_2]$,

$$|g(\sigma + it)| \le \omega_1(\sigma, T, \alpha) e^{-\alpha\left(\frac{|t|}{T}\right)^\beta} \quad \text{for all } t, \tag{2.12}$$

$$\omega_2(\sigma, T, \alpha) \le |g(\sigma + it)| \quad \text{for all } t \in [H, T]. \tag{2.13}$$

In addition, we assume that $|g|$ is even in t:

$$|g(\sigma - it)| = |g(\sigma + it)| \text{ for } \sigma \in (\sigma_1, \sigma_2) \text{ and } t \in \mathbb{R}. \tag{2.14}$$

Thus $F_X(\sigma, T) \ll_g \mathcal{M}_{g,T}(X,\sigma)$, and more precisely

$$F_X(\sigma, T) \le \frac{\mathcal{M}_{g,T}(X,\sigma)}{2(\omega_2(\sigma, T, \alpha))^2}. \tag{2.15}$$

In this article, we shall choose a family of weights of the form

$$g(s) = g_T(s) = \frac{s-1}{s} e^{\alpha(\frac{s}{T})^2}, \text{ where } \alpha > 0. \tag{2.16}$$

These weights will satisfy the above conditions with $\beta = 2$. We remark that Ingham [12] made use of the weight $g(s) = \frac{s-1}{s\cos(\frac{1}{2T})}$ and Ramaré [24] used $g(s) = \frac{s-1}{s(\cos s)^{\frac{1}{2T}}}$. These weights satisfy (2.12) with $\beta = 1$. We also studied the weights $g(s) = \frac{s-1}{s(\cos s)^{\frac{\alpha}{T}}}$ and $g(s) = \frac{s-1}{s(\cos\frac{\alpha}{T})}$. However, we obtained the best results with g given by (2.16). The functions g are chosen so that for fixed σ, $g(\sigma+it)$ behave likes the indicator function, $\mathbb{1}_{[0,T]}(t)$, and for t large, $g(\sigma+it)$ has rapid decay. Nevertheless, it is an open problem to determine the best weights g to use in this problem.

2.4. **Final bound.** Finally, to bound the integral $\mathcal{M}_{g,T}$, we appeal to a convexity estimate for integrals (see [7]). For $\sigma_2 > 1$ (and σ_2 close to 1), if $\frac12 \le \sigma \le \sigma_2$, then

$$\mathcal{M}_{g,T}(X,\sigma) \le \mathcal{M}_{g,T}(X,\tfrac12)^{\frac{\sigma_2-\sigma}{\sigma_2-\frac12}}\mathcal{M}_{g,T}(X,\sigma_2)^{\frac{\sigma-\frac12}{\sigma_2-\frac12}}. \tag{2.17}$$

The largest contribution arises from $\mathcal{M}_{g,T}(X,\frac12)$. To bound this we make use of:

- bounds (2.12), (2.13) for g (see Lemma 3.7),
- a version of Montgomery and Vaughan's Mean Value Theorem for Dirichlet polynomials (see Lemma 3.6),
- bounds for arithmetic sums to bound the second moment of the mollifier M_X (we use Ramaré's bounds, see Lemma 3.3 and 3.4),
- the most recent explicit subconvexity bound for the Riemann zeta function (due to Hiary [9], see Lemma 3.2).

3. Preliminary lemmas

3.1. **Bounds for the Riemann zeta function.** In this section we record a number of bounds for the zeta function. Rademacher [22, Theorem 4] established the following explicit convexity bound.

Lemma 3.1. *For $-\frac12 \le -\eta \le \sigma \le 1+\eta \le \frac32$, we have*

$$|\zeta(s)| \le 3\frac{|1+s|}{|1-s|}\left(\frac{|1+s|}{2\pi}\right)^{\frac12(1-\sigma+\eta)}\zeta(1+\eta). \tag{3.1}$$

The next lemma is an explicit version of van der Corput's subconvexity bound for ζ on the critical line, recently proven by Hiary. [9].

Lemma 3.2. *We have*

$$|\zeta(\tfrac12+it)| \le a_1 t^{\frac16}\log t \qquad \text{for all } t \ge 3, \tag{3.2}$$

$$\max_{|t|\le T}|\zeta(\tfrac12+it)| \le a_1 T^{\frac16}\log T + a_2 \qquad \text{for all } T > 0, \tag{3.3}$$

with

$$a_1 = 0.63 \text{ and } a_2 = 2.851. \tag{3.4}$$

Proof of Lemma 3.2. Statement (3.2) is [9, Theorem 1.1]. For $T \in [0,3]$, [9, Theorem 1.1] provides that $|\zeta(\frac{1}{2}+it)| \le 1.461$. We find that the minimum of the function $t^{\frac{1}{6}}\log(t)$ occurs when $t = e^{-6}$. We require the polynomial $a_1 t^{\frac{1}{6}}\log(t) + a_2 \ge 1.461$, choosing a_2 as in the statement of the lemma achieves this. □

3.2. **Bounds for arithmetic sums.** We list here some preliminary lemmas from [24] providing estimates for finite arithmetic sums. Let

$$b_1 = 0.62,\ b_2 = 1.048,\ b_3 = 0.605,\ \text{and } b_4 = 0.529. \tag{3.5}$$

Lemma 3.3. *We have*

$$\sum_{n\le X}\mu^2(n) \le b_1 X \quad \text{for all} \quad X \ge 1700, \tag{3.6}$$

$$\sum_{n\le X}\frac{\mu^2(n)}{n} - \frac{6}{\pi^2}\log X \le b_2 \quad \text{for all} \quad X \ge 1002. \tag{3.7}$$

(3.6) is [24, Lemma 3.1] and (3.7) is [24, Lemma 3.4].

Lemma 3.4. *Let $\tau > 1, \delta > 0$, $X \ge 10^9$, and γ denotes Euler's constant. Then*

$$\sum_{X<n<5X}\frac{\lambda_X(n)^2}{n^2} \le \frac{b_3}{X}, \tag{3.8}$$

$$\sum_{n\ge1}\frac{\lambda_X(n)^2}{n^\tau} \le \frac{b_4\tau^2}{\tau-1}e^{\gamma(\tau-1)}\log X, \tag{3.9}$$

$$\sum_{n\ge1}\frac{\lambda_X(n)^2}{n^{1+\frac{\delta}{\log X}}} \le \frac{b_4}{\delta}\Big(1+\frac{\delta}{\log X}\Big)^2 e^{\frac{\delta\gamma}{\log X}}(\log X)^2, \tag{3.10}$$

$$\sum_{n\ge1}\frac{\lambda_X(n)^2}{n^{2+\frac{2\delta}{\log X}}} \le \frac{b_4}{5\delta e^\delta}\Big(1+\frac{\delta}{\log X}\Big)^2 e^{\frac{\delta(\gamma-\log 5)}{\log X}}\frac{(\log X)^2}{X} + \frac{b_3 e^{-2\delta}}{X}. \tag{3.11}$$

Proof. (3.8) is [24, Lemma 5.6] and (3.9) is [24, Lemma 5.5]. (3.10) is a direct consequence of (3.9), taking $\tau = 1 + \frac{\delta}{\log X}$.
For (3.11) we set $\tau = 2 + \frac{2\delta}{\log X}$. Since $\lambda_X(n)^2 = 0$ when $1 \le n \le X$, then

$$\sum_{n\ge1}\frac{\lambda_X(n)^2}{n^\tau} = \sum_{X<n<5X}\frac{\lambda_X(n)^2}{n^\tau} + \sum_{n\ge5X}\frac{\lambda_X(n)^2}{n^\tau}.$$

Since $\tau \ge 2$, we use (3.8) and find that the first sum is

$$\le \frac{1}{X^{\tau-2}}\sum_{X<n<5X}\frac{\lambda_X(n)^2}{n^2} \le \frac{1}{X^{\tau-2}}\frac{b_3}{X} = \frac{b_3 e^{-2\delta}}{X}.$$

We bound the second sum using $n^\tau \ge (5X)^{1+\frac{\delta}{\log X}} n^{1+\frac{\delta}{\log X}}$ and (3.10). We find that it is

$$\le \frac{1}{(5X)^{1+\frac{\delta}{\log X}}}\frac{b_4}{\delta}\Big(1+\frac{\delta}{\log X}\Big)^2 e^{\frac{\delta\gamma}{\log X}}(\log X)^2.$$

Combining bounds

$$\sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^\tau}\leq \frac{b_4}{5\delta e^\delta}\Big(1+\frac{\delta}{\log X}\Big)^2 e^{\frac{\delta(\gamma-\log 5)}{\log X}}\frac{(\log X)^2}{X}+\frac{b_3e^{-2\delta}}{X}.$$

□

Lemma 3.5. *Let $\tau>1$ and γ is Euler's constant. Then for $X\geq 1$,*

$$\sum_{n\geq X}\frac{d(n)}{n^\tau}\leq \frac{\tau}{X^{\tau-1}}\left(\frac{\log X}{\tau-1}+\frac{1}{(\tau-1)^2}+\frac{\gamma}{\tau-1}+\frac{7}{12\tau X}\right) \tag{3.12}$$

and for $X\geq 47$,

$$\sum_{n\geq X}\frac{d(n)^2}{n^\tau}\leq \frac{2\tau}{X^{\tau-1}}\Big(\frac{(\log X)^3}{\tau-1}+\frac{3\log^2 X}{(\tau-1)^2}+\frac{6\log X}{(\tau-1)^3}+\frac{6}{(\tau-1)^4}\Big). \tag{3.13}$$

Proof. By partial summation, we have

$$\sum_{n\geq X}\frac{d(n)}{n^\tau}\leq \tau\int_X^\infty \frac{\sum_{n\leq t}d(n)}{t^{\tau+1}}dt.$$

Using $\sum_{n\leq t}d(n)\leq t(\log t+\gamma+\frac{7}{12t})$, for $t\geq 1$, which follows from [23, Equation 3.1], we have

$$\sum_{n\geq X}\frac{d(n)}{n^\tau}\leq \tau\left(\int_X^\infty\frac{\log t}{t^\tau}dt+\gamma\int_X^\infty\frac{dt}{t^\tau}+\frac{7}{12}\int_X^\infty\frac{dt}{t^{\tau+1}}\right).$$

By applying the integrals

$$\int_X^\infty\frac{\log t}{t^c}dt=\frac{\log X}{(c-1)X^{c-1}}+\frac{1}{(c-1)^2X^{c-1}}\text{ and }\int_X^\infty\frac{dt}{t^c}=\frac{1}{(c-1)X^{c-1}},\text{ where } c>1,$$

we obtain (3.12). The second estimate is similar. We have

$$\sum_{n\geq X}\frac{d(n)^2}{n^\tau}\leq \tau\int_X^\infty \frac{\sum_{n\leq t}d(n)^2}{t^{\tau+1}}\,dt.$$

It suffices to use the elementary bound $\sum_{n\leq t}d(n)^2\leq t(\log t+1)^3\leq 2t\log^3 t$ for $t\geq 47$, derived by Gowers [6]. Thus

$$\sum_{n\geq X}\frac{d(n)^2}{n^\tau}\leq 2\tau\int_X^\infty\frac{\log^3 t}{t^\tau}\,dt=2\tau\left(\frac{\frac{(\log X)^3}{\tau-1}+\frac{3\log^2 X}{(\tau-1)^2}+\frac{6\log X}{(\tau-1)^3}+\frac{6}{(\tau-1)^4}}{X^{\tau-1}}\right).$$

□

3.3. **Mean value theorem for Dirichlet polynomials.** We require Montgomery and Vaughan's mean value theorem for Dirichlet polynomials in the form derived by Ramaré [24].

Lemma 3.6. *Let (u_n) be a real-valued sequence. For every $T\geq 0$ we have*

$$\int_0^T\Big|\sum_{n=1}^\infty u_n n^{it}\Big|^2dt\leq \sum_{n\geq 1}|u_n|^2(T+\pi m_0(n+1)), \tag{3.14}$$

with

$$m_0=\sqrt{1+\frac{2}{3}\sqrt{\frac{6}{5}}}. \tag{3.15}$$

Let $0 < T_1 < T_2$. *Then*

$$\int_{T_1}^{T_2} |\sum_{n=1}^{\infty} u_n n^{it}|^2 dt \le \sum_{n\ge 1} |u_n|^2 (T_2 - T_1 + 2\pi m_0(n+1)). \tag{3.16}$$

Proof. The inequality (3.14) is [24, Lemma 6.5], and (3.16) follows by the same proof. This argument is an explicit version of Corollary 3 of [18] which makes use of the main theorem of [21]. Note that (3.16) follows from two applications of (3.14). □

3.4. Choice for the smooth weight g.

Lemma 3.7. *Let* $\alpha > 0$ *and* $\beta = 2$. *Let* $s = \sigma + it$ *and let* g *be as defined in* (2.16)*:*

$$g(s) = \frac{s-1}{s} e^{\alpha\left(\frac{s}{T}\right)^2}. \tag{3.17}$$

Let $\sigma_1 = \frac{1}{2}, \sigma_2 > 1$, *and* $H < T$. *Define*

$$\omega_1(\sigma, T, \alpha) = e^{\alpha\left(\frac{\sigma}{T}\right)^2}, \tag{3.18}$$

$$\omega_2(\sigma, T, \alpha) = \left(1 - \frac{1}{H}\right) e^{\alpha\left(\frac{\sigma}{T}\right)^2 - \alpha}. \tag{3.19}$$

Then for $\frac{1}{2} \le \sigma \le \sigma_2$, g *satisfies* (2.12) *and* (2.13)*:*

$$|g(\sigma + it)| \le \omega_1(\sigma, T, \alpha) e^{-\alpha\left(\frac{|t|}{T}\right)^2} \quad \textit{for all } t, \tag{3.20}$$

$$\omega_2(\sigma, T, \alpha) \le |g(\sigma + it)| \quad \textit{for } H \le t \le T. \tag{3.21}$$

Proof. Since $\sigma \ge \frac{1}{2}$, we have $\left|\frac{s-1}{s}\right|^2 = 1 - \frac{2\sigma-1}{\sigma^2+t^2} \le 1$. Thus $|g(s)| \le |e^{\alpha\left(\frac{s}{T}\right)^2}| = e^{\frac{\alpha\sigma^2}{T^2}} e^{\frac{-\alpha t^2}{T^2}}$ and we have the expression for $\omega_1(\sigma, T, \alpha)$.
In addition, $|\frac{s-1}{s}| = |1 - \frac{1}{s}| \ge 1 - \frac{1}{|s|} \ge 1 - \frac{1}{|t|}$, so for all $t \in [H, T]$, we have

$$|g(s)| \ge \left(1 - |t|^{-1}\right) e^{\frac{\alpha\sigma^2}{T^2}} e^{\frac{-\alpha t^2}{T^2}} \ge (1 - H^{-1}) e^{\frac{\alpha\sigma^2}{T^2}} e^{-\alpha},$$

which gives $\omega_2(\sigma, T, \alpha)$. □

4. Proof of the Main Theorem

Unless specified in the rest of the article, we set $H_0 = 3.0610046 \cdot 10^{10}$ and we have the following conditions on the parameters k, σ_1, δ, and σ_2:

$$k \ge \frac{10^9}{H_0}, \ \sigma_1 = \frac{1}{2}, \ \delta > 0, \text{ and } \sigma_2 = 1 + \frac{\delta}{\log X}. \tag{4.1}$$

4.1. Bounding $F_X(\sigma, T)$.

We establish here some preliminary lemmas to estimate $F_X(\sigma, T)$ at $\frac{1}{2}$ and at $1 + \frac{\delta}{\log X}$.

4.1.1. *Bounding $F_X(\frac12,T)$.* We first need to bound the second moment of $M_X(\frac12+it)$, where M_X is defined in (2.4).

Lemma 4.1. *Let $T>0$, $X\ge kH_0$, and k satisfies* (4.1). *Then*

$$\int_0^T \left|M_X(\tfrac12+it)\right|^2 dt \le (C_1T+C_2X)(\log X), \tag{4.2}$$

where

$$C_1=C_1(k)=\frac{6}{\pi^2}+\frac{b_2}{\log(kH_0)}, \tag{4.3}$$

$$C_2=C_2(k)=\frac{\pi m_0 b_1}{\log(kH_0)}+\frac{6m_0}{\pi kH_0}+\frac{\pi m_0 b_2}{kH_0\log(kH_0)}, \tag{4.4}$$

and the b_i's are defined in (3.5) *and m_0 in* (3.15).

Proof. We apply (3.14) to $u_n=\frac{\mu(n)}{n^{\frac12}}$:

$$\int_0^T \left|M_X\left(\tfrac12+it\right)\right|^2 dt \le \sum_{n\le X}\frac{\mu^2(n)}{n}(T+\pi m_0(n+1)).$$

Since $X\ge 1700$, we apply (3.6) to $(T+\pi m_0)\sum_{n\le X}\frac{\mu^2(n)}{n}$ and (3.7) to $(\pi m_0)\sum_{n\le X}\mu^2(n)$ respectively. We factor $\log X$ to give

$$\begin{aligned}\int_0^T \left|M_X\left(\tfrac12+it\right)\right|^2 dt &= (T+\pi m_0)\left(\frac{6}{\pi^2}\log X+b_2\right)+\pi m_0 b_1 X\\ &=\left(\left(\frac{6}{\pi^2}+\frac{b_2}{\log X}\right)T+\left(\frac{6m_0}{\pi X}+\frac{\pi m_0 b_2}{X\log X}+\frac{\pi m_0 b_1}{\log X}\right)X\right)(\log X),\end{aligned}$$

and use the fact that $X\ge kH_0$ to obtain the announced bound. □

Lemma 4.2. *Let $T>0$, $X\ge kH_0$, and k satisfies* (4.1). *Then*

$$F_X(\tfrac12,T)\le C_4\left(T^{\frac16}\log T+\frac{a_2}{a_1}\right)^2\left(T+\frac{C_2}{C_1}X\right)(\log X), \tag{4.5}$$

where a_1,a_2 are defined in (3.4), *C_1 in* (4.3), *C_2 in* (4.4), *and*

$$a_3=-\frac{6a_1}{e}+a_2, \tag{4.6}$$

$$C_3=C_3(k)=a_3^2C_1(k)\log(kH_0), \tag{4.7}$$

$$C_4=C_4(k)=C_1(k)a_1^2\left(1+\frac{1}{\sqrt{C_3(k)}}\right)^2. \tag{4.8}$$

Proof. We have from the definition of $F_X(\sigma,T)$ given as (2.10) and Minkowski's inequality that

$$\sqrt{|F_X(\tfrac12,T)|}\le\sqrt{\int_0^T|\zeta(\tfrac12+it)M_X(\tfrac12+it)|^2dt}+\sqrt{T}.$$

To the last integral we apply Hiary's subconvexity bound (3.3) to bound zeta and (4.2) to bound the mean square of M_X. We let I_0 denote the resulting bound so that

$$I_0=(a_1T^{\frac16}\log T+a_2)^2(C_1T+C_2X)(\log X),$$

and thus

$$|F_X(\tfrac{1}{2},T)| \leq \left(\sqrt{I_0}+\sqrt{T}\right)^2 = I_0\left(1+\sqrt{\frac{T}{I_0}}\right)^2.$$

We note that $a_1T^{\frac{1}{6}}\log T + a_2$ is minimized at $T=e^{-6}$ and we let a_3 represent this minimum. Then

$$I_0 \geq a_3^2(C_1T+C_2X)\log X \geq a_3^2 C_1 T\log X.$$

We conclude with the lower bound $\frac{I_0}{T} \geq a_3^2 C_1 \log(kH_0)$, which is labeled C_3, and

$$I_0 = C_1a_1^2\left(T^{\frac{1}{6}}\log T+\frac{a_2}{a_1}\right)^2\left(T+\frac{C_2}{C_1}X\right)(\log X),$$

which completes the proof. □

4.1.2. *Bounding $F_X(\sigma_2,T)$ at $\sigma_2 = 1+\frac{\delta}{\log X}$.*

Lemma 4.3. *Let $T>0$, $X\geq kH_0$ and k,δ,σ_2 satisfy* (4.1)*. Then*

$$F_X(\sigma_2,T) \leq \left(C_5(k,\delta)+\frac{C_6(k,\delta)(T+\pi m_0)}{X}\right)(\log X)^2, \tag{4.9}$$

where

$$C_5(k,\delta) = \frac{\pi m_0 b_4}{2\delta}\Big(1+\frac{2\delta}{\log(kH_0)}\Big)^2 e^{\frac{2\delta\gamma}{\log(kH_0)}}, \tag{4.10}$$

$$C_6(k,\delta) = \frac{b_4}{5\delta e^{\delta}}\Big(1+\frac{\delta}{\log(kH_0)}\Big)^2+\frac{b_3e^{-2\delta}}{(\log(kH_0))^2}, \tag{4.11}$$

the b_i's are defined in (3.5)*, m_0 in* (3.15)*, and γ is Euler's constant.*

Proof. Recall that F_X is defined by (2.10) and by (2.6) we have

$$F_X(\sigma_2,T) = \int_0^T |f_X(\sigma_2+it)|^2dt = \int_0^T \Big|\sum_{n\geq 1}\frac{\lambda_X(n)}{n^{\sigma_2+it}}\Big|^2dt.$$

Inequality (3.14) implies the bound

$$F_X(\sigma_2,T) \leq \pi m_0\sum_{n\geq1}\frac{\lambda_X(n)^2}{n^{2\sigma_2-1}}+(T+\pi m_0)\sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^{2\sigma_2}}.$$

For $2\sigma_2-1 = 1+\frac{2\delta}{\log X}$ and $2\sigma_2 = 2+\frac{2\delta}{\log X}$, we apply the bounds for arithmetic sums (3.10) and (3.11) to respectively bound the two above sums. Thus

$$\sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^{1+2\frac{\delta}{\log X}}} \leq \frac{b_4}{2\delta}\Big(1+\frac{2\delta}{\log X}\Big)^2 e^{\frac{2\delta\gamma}{\log X}}(\log X)^2,$$

$$\text{and}\quad \sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^{2+\frac{2\delta}{\log X}}} \leq \frac{b_4}{5\delta e^{\delta}}\Big(1+\frac{\delta}{\log X}\Big)^2\frac{(\log X)^2}{X}+\frac{b_3e^{-2\delta}}{X}.$$

We combine these results and use the fact that $X\geq kH_0$ to complete the proof. □

From here we may derive a bound for $\mathcal{M}_{g,T}(X,\sigma)$.

4.2. **Explicit upper bounds for the mollifier $\mathcal{M}_{g,T}(X,\sigma)$.** The results in this section are proven for a general weight g satisfying the conditions described in Section 2.3. In [7, Theorem 7], Hardy et al. proved the following convexity estimate:

Lemma 4.4. *Let $\frac{1}{2} \le \sigma_1 < 1 < \sigma_2$, let $T > 0$, and $X > 1$. Then*

$$\mathcal{M}_{g,T}(X,\sigma) \le \mathcal{M}_{g,T}(X,\sigma_1)^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}} \mathcal{M}_{g,T}(X,\sigma_2)^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}. \tag{4.12}$$

In order to obtain a bound for the mollifier $\mathcal{M}_{g,T}(X,\sigma)$ inside the strip $\frac{1}{2} \le \sigma \le 1 + \frac{\delta}{\log X}$, we need explicit bounds at the extremities $\frac{1}{2}$ and $1 + \frac{\delta}{\log X}$.

Lemma 4.5. *Let $T > 0$, $X > 0, \sigma \ge \frac{1}{2}$, and let g satisfy conditions* (2.12) *and* (2.14)*. Then*

$$\mathcal{M}_{g,T}(X,\sigma) \le 4\omega_1(\sigma,T,\alpha)^2 \alpha\beta \int_0^\infty x^{\beta-1} e^{-2\alpha x^\beta} F_X(\sigma, xT) dx. \tag{4.13}$$

Proof. By (2.14) and $|g(\sigma+it)| = |g(\sigma-it)|$ for $t \in \mathbb{R}$ and by an application of (2.12) to the weight g in the definition (2.11) of $\mathcal{M}_{g,T}(X,\sigma)$, we have

$$\mathcal{M}_{g,T}(X,\sigma) \le 2\omega_1(\sigma,T,\alpha)^2 \int_0^\infty e^{-2\alpha\left(\frac{t}{T}\right)^\beta} |f_X(\sigma+it)|^2 dt. \tag{4.14}$$

Note that $\int_0^U |f_X(\sigma+it)|^2 dt = F_X(\sigma,U)$ with $F_X(\sigma,0) = 0$ and $\lim_{U\to\infty}\left(F_X(\sigma,U)e^{-2\alpha\left(\frac{U}{T}\right)^\beta}\right) = 0$. Integrating by parts gives

$$\begin{aligned}\int_0^\infty e^{-2\alpha\left(\frac{t}{T}\right)^\beta} |f_X(\sigma+it)|^2 dt &= 2\alpha\beta \int_0^\infty \left(\frac{t}{T}\right)^\beta e^{-2\alpha\left(\frac{t}{T}\right)^\beta} F_X(\sigma,t)\frac{dt}{t} \\ &= 2\alpha\beta \int_0^\infty x^\beta e^{-2\alpha x^\beta} F_X(\sigma,xT)\frac{dx}{x},\end{aligned}$$

by the variable change $x = \frac{t}{T}$. This combined with (4.14) yields the announced (4.13). $\square$

4.2.1. *Bounding $\mathcal{M}_{g,T}(X,\frac{1}{2})$.* Let $\alpha, \beta, A > 0$ and let n be a non-negative integer. We define

$$I(A,n) = \int_0^\infty x^A e^{-2\alpha x^\beta} (\log x)^n dx. \tag{4.15}$$

In our context, $I(A,n)$ is a constant depending on parameters A and n and is $\mathcal{O}(1)$ in comparison with T. The change of variable $y = 2\alpha x^B$ leads to the identity

$$I(A,n) = (2\alpha)^{-\frac{A+1}{\beta}} \beta^{-(n+1)} \sum_{j=0}^n \binom{n}{j} (-\log(2\alpha))^j \Gamma^{(n-j)}\left(\frac{A+1}{\beta}\right), \tag{4.16}$$

where $\Gamma^{(j)}(z)$ denotes the j-th derivative of Euler's gamma function. We also define

$$\begin{aligned}\mathcal{J}(k,T) =& I(\beta+\tfrac{1}{3},0) + \frac{C_2}{C_1}kI(\beta-\tfrac{2}{3},0) + \frac{2I(\beta+\frac{1}{3},1) + 2\frac{C_2}{C_1}kI(\beta-\frac{2}{3},1)}{(\log T)} \\ &+ \frac{I(\beta+\frac{1}{3},2) + \frac{C_2}{C_1}kI(\beta-\frac{2}{3},2)}{(\log T)^2} + \frac{2a_2\left(I(\beta+\frac{1}{6},0) + \frac{C_2k}{C_1}I(\beta-\frac{5}{6},0)\right)}{a_1T^{\frac{1}{6}}(\log T)} \\ &+ \frac{2a_2\left(I(\beta+\frac{1}{6},1) + \frac{C_2k}{C_1}I(\beta-\frac{5}{6},1)\right)}{a_1T^{\frac{1}{6}}(\log T)^2} + \frac{a_2^2\left(I(\beta,0) + \frac{C_2k}{C_1}I(\beta-1,0)\right)}{a_1^2T^{\frac{1}{3}}(\log T)^2},\end{aligned} \tag{4.17}$$

$$\mathcal{U}(\alpha,k,T)=4\alpha\beta C_4\omega_1(\tfrac{1}{2},T,\alpha)^2\mathcal{J}(k,T), \tag{4.18}$$

where ω_1 and C_4 are respectively defined in (3.18) and (4.8). We remark that in the case of our weight g, we have $\beta=2$. Thus in our calculations of $\mathcal{J}(k,T)$ we specialize to $\beta=2$.

Lemma 4.6. *Let $\alpha,\beta>0$ and g be a function satisfying* (2.12) *and* (2.14)*. Let $T\geq H_0$, $X=kT$, and k satisfies* (4.1)*. Then*

$$\mathcal{M}_{g,T}(X,\tfrac{1}{2})\leq \mathcal{U}(\alpha,k,T)(\log(kT))(\log T)^2T^{\frac{4}{3}}.$$

Proof. We combine the bound (4.13) for $\mathcal{M}_{g,T}$ with the bound (4.5) for $F_X(\frac{1}{2},xT)$:

$$\begin{aligned}\mathcal{M}_{g,T}(X,\tfrac{1}{2})\leq 4\alpha\beta C_4\omega_1(\tfrac{1}{2},T,\alpha)^2(\log X)\Big\{&T^{\frac{4}{3}}\int_0^\infty x^{\beta+\frac{1}{3}}(\log(xT))^2e^{-2\alpha x^\beta}dx\\ &+\frac{2a_2}{a_1}T^{\frac{7}{6}}\int_0^\infty x^{\beta+\frac{1}{6}}\log(xT)e^{-2\alpha x^\beta}dx+\frac{a_2^2}{a_1^2}T\int_0^\infty x^\beta e^{-2\alpha x^\beta}dx\\ &+\frac{C_2}{C_1}XT^{\frac{1}{3}}\int_0^\infty x^{\beta-\frac{2}{3}}(\log(xT))^2e^{-2\alpha x^\beta}dx+\frac{2a_2}{a_1}\frac{C_2}{C_1}XT^{\frac{1}{6}}\int_0^\infty x^{\beta-\frac{5}{6}}\log(xT)e^{-2\alpha x^\beta}dx\\ &+\frac{a_2^2}{a_1^2}\frac{C_2}{C_1}X\int_0^\infty x^{\beta-1}e^{-2\alpha x^\beta}dx\Big\}.\end{aligned}$$

We also use the fact that $(\log(xT))^2=(\log x)^2+2(\log x)(\log T)+(\log T)^2$ and obtain

$$\begin{aligned}\mathcal{M}_{g,T}(X,\tfrac{1}{2})\leq 4\alpha\beta C_4\omega_1(\tfrac{1}{2},T,\alpha)^2(\log X)\Big\{&T^{\frac{4}{3}}\big(I(\beta+\tfrac{1}{3},2)+2(\log T)I(\beta+\tfrac{1}{3},1)\\ &+(\log T)^2I(\beta+\tfrac{1}{3},0)\big)+\frac{2a_2}{a_1}T^{\frac{7}{6}}\big(I(\beta+\tfrac{1}{6},1)+(\log T)I(\beta+\tfrac{1}{6},0)\big)+\frac{a_2^2}{a_1^2}TI(\beta,0)\\ &+\frac{C_2}{C_1}XT^{\frac{1}{3}}\big(I(\beta-\tfrac{2}{3},2)+2(\log T)I(\beta-\tfrac{2}{3},1)+(\log T)^2I(\beta-\tfrac{2}{3},0)\big)\\ &+\frac{2a_2}{a_1}\frac{C_2}{C_1}XT^{\frac{1}{6}}\big(I(\beta-\tfrac{5}{6},1)+(\log T)I(\beta-\tfrac{5}{6},0)\big)+\frac{a_2^2}{a_1^2}\frac{C_2}{C_1}XI(\beta-1,0)\Big\},\end{aligned}$$

where I is the integral defined in (4.15). At this point we choose $X=kT$ so as to optimize the above bound, and we factor out the main term $T^{\frac{4}{3}}(\log T)^2$:

$$\begin{aligned}\mathcal{M}_{g,T}(X,\tfrac{1}{2})\leq 4\alpha\beta C_4\omega_1(\tfrac{1}{2},T,\alpha)^2(\log(kT))(\log T)^2T^{\frac{4}{3}}\Big\{&I(\beta+\tfrac{1}{3},0)+\frac{kC_2}{C_1}I(\beta-\tfrac{2}{3},0)\\ +2\frac{I(\beta+\frac{1}{3},1)+\frac{kC_2}{C_1}I(\beta-\frac{2}{3},1)}{(\log T)}+\frac{I(\beta+\frac{1}{3},2)+\frac{kC_2}{C_1}I(\beta-\frac{2}{3},2)}{(\log T)^2}&+\frac{2a_2}{a_1}\frac{I(\beta+\frac{1}{6},0)+\frac{kC_2}{C_1}I(\beta-\frac{5}{6},0)}{(\log T)T^{\frac{1}{6}}}\\ +\frac{2a_2}{a_1}\frac{I(\beta+\frac{1}{6},1)+\frac{kC_2}{C_1}I(\beta-\frac{5}{6},1)}{(\log T)^2T^{\frac{1}{6}}}&+\frac{a_2^2}{a_1^2}\frac{I(\beta,0)+\frac{kC_2}{C_1}I(\beta-1,0)}{(\log T)^2T^{\frac{1}{3}}}\Big\}.\end{aligned}$$

We recognize in the above term between brackets $\mathcal{J}(k,T)$ as introduced in (4.17). □

4.2.2. *Bounding $\mathcal{M}_{g,T}(X,\sigma_2)$ at $\sigma_2=1+\frac{\delta}{\log X}$.*

Lemma 4.7. *Let g be as defined in Lemma 3.7. Let $T\geq H_0$, $X=kT$, and k,δ,σ_2 satisfy* (4.1)*. Then*

$$\mathcal{M}_{g,T}(X,\sigma_2)\leq \mathcal{V}(\alpha,k,\delta,T)(\log(kT))^2,$$

where

$$\mathcal{V}(\alpha,k,\delta,T) = 8\alpha\omega_1(\sigma_2,T,\alpha)^2\mathcal{K}(k,\delta,T), \tag{4.19}$$

$$\mathcal{K}(k,\delta,T) = \left(C_5(k,\delta) + \frac{C_6(k,\delta)\pi m_0}{kT}\right) I(1,0) + \frac{C_6(k,\delta)}{k} I(2,0), \tag{4.20}$$

and $m_0, \omega_1, C_5, C_6,$ *and* I *are respectively defined in* (3.15), (3.18), (4.10), (4.11), *and* (4.15).

Proof. We combine the bound (4.13) for $\mathcal{M}_{g,T}$ with the bound (4.9) for $F_X(\sigma_2, xT)$ (since $X \geq kH_0$) to obtain

(4.21)

$$\mathcal{M}_{g,T}(X,\sigma_2) \leq 4\alpha\beta\omega_1(\sigma_2,T,\alpha)^2\left(\int_0^\infty x^{\beta-1}e^{-2\alpha x^\beta}\left(C_5(k,\delta)+\frac{C_6(k,\delta)(xT+\pi m_0)}{X}\right)(\log X)^2 dx\right).$$

Rearranging this and recalling the definition for I in (4.15) we obtain

$$\begin{aligned}\mathcal{M}_{g,T}(X,\sigma_2) \leq 4\alpha\beta\omega_1(\sigma_2,T,\alpha)^2(\log X)^2 \Bigg(\left(C_5(k,\delta) + \frac{C_6(k,\delta)\pi m_0}{X}\right) I(\beta-1,0) \\ + \frac{C_6(k,\delta)T}{X} I(\beta,0)\Bigg).\end{aligned}$$

We conclude by noting that $X = kT$ and for our g, $\beta = 2$. □

4.2.3. *Conclusion.* Finally, we provide bounds for $\mathcal{M}_{g,T}$.

Lemma 4.8. *Let* g *be as defined in Lemma 3.7. Let* $T \geq H_0$, $X = kT$, *and* k *satisfies* (4.1). *Assume* $\frac{1}{2} \leq \sigma \leq 1 + \frac{\delta}{\log X}$. *Then*

$$\begin{aligned}\mathcal{M}_{g,T}(X,\sigma) \leq e^{\frac{8}{3}\delta(2\sigma-1)M(k,\sigma)+\frac{4\delta(2\sigma-1)\log\log H_0}{\log(kH_0)+2\delta}}\,\mathcal{U}(\alpha,k,T)^{2(1-\sigma)+\frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}} \times \\ \mathcal{V}(\alpha,k,\delta,T)^{2\sigma-1-\frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}}(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)}T^{\frac{8}{3}(1-\sigma)},\end{aligned} \tag{4.22}$$

where $\mathcal{U}$ *and* $\mathcal{V}$ *are respectively defined in* (4.18) *and* (4.19) *and*

$$M(k,\delta) = \max\left(\frac{\log H_0}{\log(kH_0)+2\delta}, 1\right). \tag{4.23}$$

Proof. Let $\sigma_1 = \frac{1}{2}$ andf $\sigma_2 = 1 + \frac{\delta}{\log X}$ and $\sigma \in [\sigma_1, \sigma_2]$. We apply the convexity inequality (4.12) with exponents

$$a = \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1} = \frac{1 + \frac{\delta}{\log X} - \sigma}{(1 + \frac{\delta}{\log X}) - \frac{1}{2}} \text{ and } b = 1 - a = \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1} = \frac{\sigma - \frac{1}{2}}{(1 + \frac{\delta}{\log X}) - \frac{1}{2}} \tag{4.24}$$

in combination with Lemmas 4.6, Lemma 4.7 to obtain

$$\mathcal{M}_{g,T}(X,\sigma) \leq \mathcal{U}(\alpha,k,T)^a\mathcal{V}(\alpha,k,\delta,T)^b(\log(kT))^{a+2b}(\log T)^{2a}T^{\frac{4}{3}a}. \tag{4.25}$$

Next, from the definitions of (4.24) it may be checked that

$$a = 2(1-\sigma) + \frac{2\delta(2\sigma-1)}{\log X + 2\delta}, \text{ and } b = 2\sigma - 1 - \frac{2\delta(2\sigma-1)}{\log X + 2\delta}. \tag{4.26}$$

From these equalities it follows that $a+2b\le 2\sigma$. Using (4.26) and the bound for $a+2b$ (since $\log(kT)\ge\log(kH_0)\ge\log(10^9)>1$), we have

$$\mathcal{M}_{g,T}(X,\sigma)\le e^{\frac{4}{3}\times\frac{2\delta(2\sigma-1)\log T}{\log(kT)+2\delta}+2\times\frac{2\delta(2\sigma-1)\log\log T}{\log(kT)+2\delta}}\mathcal{U}(\alpha,k,T)^{2(1-\sigma)+\frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}}\times \mathcal{V}(\alpha,k,\delta,T)^{2\sigma-1-\frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}}(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)}T^{\frac{8}{3}(1-\sigma)}. \tag{4.27}$$

Next we observe that the function $\frac{\log T}{\log(kT)+2\delta}$ decreases if $\log k+2\delta<0$ and increases if $\log k+2\delta>0$ and thus

$$\frac{\log T}{\log(kT)+2\delta}\le M(k,\delta):=\begin{cases}\frac{\log H_0}{\log(kH_0)+2\delta} & \text{if } \log k+2\delta<0,\\ 1 & \text{if } \log k+2\delta\ge 0\end{cases} \tag{4.28}$$

where $M(k,\delta)$ was defined in (4.23). Furthermore, it may be checked by the conditions on k, that $\frac{\log\log T}{\log(kT)+2\delta}$ decreases as long as $0<\delta<\frac{\log(H_0)(\log\log H_0-1)}{2}$. Using these observations in (4.27) we deduce (4.22). □

4.3. Bounding $F_X(\sigma,T)-F_X(\sigma,H)$.

Lemma 4.9. *Let g be as defined in Lemma 3.7. Let $\sigma\in[\frac{1}{2},1]$ and $\alpha>0$. Let $T\ge H_0\ge H>0$, $X=kT$, k satisfies (4.1), and $0<\delta<\frac{\log(H_0)(\log\log H_0-1)}{2}=26.36\ldots$. Then*

$$F_X(\sigma,T)-F_X(\sigma,H)\le\frac{e^{\frac{8}{3}\delta(2\sigma-1)M(k,\delta)+\frac{4\delta(2\sigma-1)\log\log H_0}{\log(kH_0)+2\delta}}\mathcal{U}(\alpha,k,T)^{2(1-\sigma)+\frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}}\mathcal{V}(\alpha,k,\delta,T)^{2\sigma-1-\frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}}}{2(\omega_2(\sigma,T,\alpha))^2}\times(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)}T^{\frac{8}{3}(1-\sigma)}, \tag{4.29}$$

where $\omega_2,\mathcal{U},\mathcal{V}$ are respectively defined in (3.19), (4.18), (4.19).

Proof. By the assumed lower bound on g, (2.13), we have

$$F_X(\sigma,T)-F_X(\sigma,H)=\int_H^T|f_X(\sigma+it)|^2dt\le\frac{1}{(\omega_2(\sigma,T,\alpha))^2}\int_H^T|g(\sigma+it)|^2|f_X(\sigma+it)|^2dt.$$

Since $t\to|g(\sigma+it)f_X(\sigma+it)|$ is even, it follows that

$$F_X(\sigma,T)-F_X(\sigma,H)\le\frac{\mathcal{M}_{g,T}(X,\sigma)}{2(\omega_2(\sigma,T,\alpha))^2}$$

and we conclude by inserting the bound (4.22) for $\mathcal{M}_{g,T}(X,\sigma)$. □

4.4. **Explicit upper bounds for $\int_{\sigma'}^{\mu}\arg h_X(\tau+iT)d\tau-\int_{\sigma'}^{\mu}\arg h_X(\tau+iH)d\tau$.** The following Proposition and Corollary are a variant of Titchmarsh [25, Lemma, p. 213]. This proposition gives a bounds for $\arg f(\sigma+iT)$ where f is a holomorphic function. The argument we use here is due to Backlund [1] in the case that $f(s)=\zeta(s)$. The cases of Dirichlet L-functions and Dedekind zeta functions have been worked out by McCurley [17] and by the first and third authors [16] respectively.

Proposition 4.10. *Let $\eta > 0$. Let $f(s)$ be a holomorphic function, for $\mathfrak{Re}(s) \ge -\eta$, real for real s. Assume there exist positive constants M and m such that*

$$|f(s)| \le M \text{ for } \mathfrak{Re}(s) \ge 1+\eta, \tag{4.30}$$

$$|\mathfrak{Re} f(1+\eta+it)| \ge m > 0 \text{ for all } t \in \mathbb{R}. \tag{4.31}$$

Let $\sigma \in (0, 1+\eta]$ and assume that U is not the ordinate of a zero of $f(s)$. Then there exists an increasing sequence of natural numbers $\{N_k\}_{k=1}^{\infty}$ such that

$$|\arg f(\sigma+iU)| \le \frac{\pi}{\log 2}\mathscr{L}_k + \frac{\pi \log M}{2\log 2} - \frac{\pi \log m}{\log 2} + \frac{\pi}{2} + o_k(1) \tag{4.32}$$

where

$$\mathscr{L}_k = \frac{1}{2\pi N_k}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log\Big(\frac{1}{2}\sum_{j=0}^{1}|f(1+\eta+(1+2\eta)e^{i\theta}+(-1)^j iU)|^{N_k}\Big)\,d\theta \tag{4.33}$$

and $o_k(1)$ is a term that approaches 0 as $k \to \infty$.

Proof of Proposition 4.10. Let $\eta > 0$. We define $\arg f(1+\eta) = 0$, and $\arg f(s) = \arctan \frac{\mathfrak{Im} f(s)}{\mathfrak{Re} f(s)}$ for $\mathfrak{Re}(s) = 1+\eta$, since, by (4.31), $\mathfrak{Re}(f(s))$ does not vanish on $\mathfrak{Re}(s) = 1+\eta$. It follows that

$$|\arg f(1+\eta+iU)| < \frac{\pi}{2}. \tag{4.34}$$

Recall that $\arg f(\sigma+iU)$ is defined by continuous variation, moving along the line $\mathcal{C}$ from $1+\eta+iU$ to $\sigma+iU$. It follows that

$$|\arg f(\sigma+iU)| \le |\Delta_{\mathcal{C}} \arg f(s)| + \frac{\pi}{2}. \tag{4.35}$$

We now bound the argument change on $\mathcal{C}$. Let $N \in \mathbb{N}$ and let

$$F_N(w) = \frac{1}{2}(f(w+iU)^N + f(w-iU)^N). \tag{4.36}$$

Since $f(s)$ is real when s is real, the reflection principle gives $F_N(\sigma) = \mathfrak{Re}\, f(\sigma+iU)^N$ for all σ real. Suppose $F_N(\sigma)$ has n real zeros in the interval $[\sigma, 1+\eta]$. These zeros partition the interval into $n+1$ subintervals. On each of these subintervals $\arg f(\sigma+iU)^N$ can change by at most π, since $\mathfrak{Re}\, f(\sigma+iU)^N$ is nonzero on the interior of each subinterval. It follows that

$$|\Delta_{\mathcal{C}} \arg f(s)| = \frac{1}{N}|\Delta_{\mathcal{C}} \arg f(s)^N| \le \frac{(n+1)\pi}{N}. \tag{4.37}$$

We now provide an upper bound for n. Jensen's theorem asserts that

$$\log|F_N(1+\eta)| + \int_0^{1+2\eta}\frac{n(u)du}{u} = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \log|F_N(1+\eta+(1+2\eta)e^{i\theta}|d\theta,$$

where $n(u)$ denotes the number of zeros of $F_N(z)$ in the circle centered at $1+\eta$ of radius u. Observe that $n(u) \ge n$ for $u \ge \frac{1}{2}+\eta$ and thus

$$n\log 2 \le \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \log|F_N(1+\eta+(1+2\eta)e^{i\theta}|d\theta - \log|F_N(1+\eta)|. \tag{4.38}$$

Trivially from (4.36),

$$|F_N(1+\eta+(1+2\eta)e^{i\theta})| \le \frac{1}{2}\sum_{j=0}^{1} |f(1+\eta+(1+2\eta)e^{i\theta}+(-1)^j iU)|^N,$$

so for the left part of the contour in (4.38),

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log|F_N(1+\eta+(1+2\eta)e^{i\theta}|d\theta \le \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log\Big(\frac{1}{2}\sum_{j=0}^{1} |f(1+\eta+(1+2\eta)e^{i\theta}+(-1)^j iU)|^N\Big)\,d\theta. \tag{4.39}$$

For the right part of the contour in (4.38), we have $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $\mathfrak{Re}(1+\eta+(1+2\eta)e^{i\theta}) \ge 1+\eta$. We apply (4.30) and obtain

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log|F_N(1+\eta+(1+2\eta)e^{i\theta}|d\theta \le \frac{N}{2}\log M. \tag{4.40}$$

To complete our bound for n, we require a lower bound for $\log|F_{N_k}(1+\eta)|$.
We write $f(1+\eta+iU) = re^{i\phi}$ and then choose (by Dirichlet's approximation theorem) an increasing sequence of positive integers N_k tending to infinity such that $N_k\phi$ tends to 0 modulo 2π. Since $\dfrac{F_{N_k}(1+\eta)}{|f(1+\eta+iU)|^{N_k}} = \dfrac{r^{N_k}\cos(N_k\phi)}{r^{N_k}}$, it follows that $\lim\limits_{k\to\infty} \dfrac{F_{N_k}(1+\eta)}{|f(1+\eta+iU)|^{N_k}} = 1$. Thus we derive

$$\log|F_{N_k}(1+\eta)| \ge N_k \log|f(1+\eta+iU)| + o_k(1),$$

where the term $o_k(1) \to 0$ as $k\to\infty$. Together with (4.31), we obtain

$$\log|F_{N_k}(1+\eta)| \ge N_k\log m + o_k(1). \tag{4.41}$$

Then (4.38), (4.39), (4.40), and (4.41) give

$$n\log 2 \le \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log\Big(\frac{1}{2}\sum_{j=0}^{1}|f(1+\eta+(1+2\eta)e^{i\theta}+(-1)^j iU)|^{N_k}\Big)\,d\theta + \frac{N_k\log M}{2} - N_k\log m + o_k(1). \tag{4.42}$$

By (4.37) it follows that

$$|\Delta_{\mathcal{C}} \arg f(s)| \le \frac{\pi}{\log 2}\mathscr{L}_k + \frac{\pi\log M}{2\log 2} - \frac{\pi\log m}{\log 2} + o_k(1),$$

where $\mathscr{L}_k$ is defined by (4.33) . We conclude by combining this with (4.35). □

We derive the following Corollary for $\arg h_X(s)$ from Proposition 4.10.

Corollary 4.11. *Let $\eta_0 = 0.23622\ldots$, $\eta\in[\eta_0,\frac{1}{2})$, and $X\ge 10^9$. Assume that $U\ge H\ge 1002$ and that U is not the ordinate of a zero of $h_X(s)$. Then for all $\tau\in(0,1+\eta]$,*

$$|\arg h_X(\tau+iU)| \le \frac{(1+2\eta)}{\log 2}\log\Big(\frac{b_8(\eta,H)}{2\pi}U\Big) + \frac{\pi(1+\eta)}{\log 2}(\log X) + \frac{\pi\log b_7(k,\eta,H_0)}{2\log 2} + \frac{\pi\log b_5(\eta)}{2\log 2} - \frac{\pi\log(1-b_6(10^9,\eta)^2)}{\log 2} + \frac{\pi}{2},$$

where b_5, b_6, b_7, b_8 are defined in (4.44), (4.45), (4.50), and (4.51).

Proof of Corollary. We apply Proposition 4.10 to $f = h_X$ as defined in (2.8):

$$h_X(s) = 1 - f_X(s)^2 = \zeta(s)M_X(s)(2 - \zeta(s)M_X(s)).$$

Let $\sigma \geq \eta + 1$ and $t \in \mathbb{R}$. We establish an upper bound for $|h_X(\sigma + it)|$. The triangle inequality in conjunction with $|\zeta(s)| \leq \zeta(1+\eta)$ and with $|M_X(s)| \leq \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^{1+\eta}} = \frac{\zeta(1+\eta)}{\zeta(2+2\eta)}$ give

$$|h_X(s)| \leq b_5(\eta) \tag{4.43}$$

with

$$b_5(\eta) = \frac{\zeta(1+\eta)^4}{\zeta(2+2\eta)^2} + \frac{2\zeta(1+\eta)^2}{\zeta(2+2\eta)}. \tag{4.44}$$

We now give a lower bound for $|\mathfrak{Re} h_X(1+\eta+it)|$. We use the reverse triangle inequality $|h_X(s)| \geq 1 - |f_X(s)|^2$. It remains to provide an upper bound for $|f_X(s)|$. Trivially from (2.5),

$$|f_X(s)| \leq \sum_{n>X} \frac{|\lambda_X(n)|}{n^{1+\eta}} \leq \sum_{n>X} \frac{d(n)}{n^{1+\eta}},$$

and by Lemma 3.5, we obtain

$$|f_X(1+\eta+it)| \leq b_6(X,\eta) = \frac{(1+\eta)(\log X)}{\eta X^{\eta}}\Big(1 + \frac{1}{\eta \log X} + \frac{\gamma}{\log X} + \frac{7\eta}{12(1+\eta)X(\log X)}\Big). \tag{4.45}$$

Note that $\frac{(\log X)}{X^{\eta}}$ decreases when $\eta > \frac{1}{\log X}$, which is the case since we assumed $\eta > \frac{1}{\log(10^9)} = 0.048254\ldots$ and $X \geq 10^9$. Thus $|f_X(s)| \leq b_6(10^9, \eta)$ and

$$|\mathfrak{Re}(h_X(s))| = |1 - \mathfrak{Re}(f_X(s))^2| \geq |1 - |f_X(s)|^2| \geq 1 - |f_X(s)|^2 \geq 1 - b_6(10^9,\eta)^2. \tag{4.46}$$

Note our assumption $\eta \geq \eta_0 = 0.23622\ldots$ ensures $1 - b_6(10^9,\eta)^2 > 0$.

Finally, we must bound $\mathscr{L}_k$ as defined in (4.33) in the case $f = h_X$. We assume w is a complex number such that $-\eta \leq \mathfrak{Re} w \leq 1+\eta$ and $|\mathfrak{Im} w| \geq U - (1+2\eta)$. Recall that by Lemma 3.1

$$|\zeta(w)| \leq 3\frac{|1+w|}{|1-w|}\Big(\frac{|w+1|}{2\pi}\Big)^{\frac{1+\eta-\mathfrak{Re} w}{2}}\zeta(1+\eta).$$

Since $\frac{|1+w|}{|1-w|} = \left|1 + \frac{2}{w-1}\right| \leq 1 + \frac{2}{|\mathfrak{Im}(w)|} \leq 1.002$ when $|\mathfrak{Im}(w)| \geq 1000$, then

$$|\zeta(w)| \leq 3.006\zeta(1+\eta)\Big(\frac{|w+1|}{2\pi}\Big)^{\frac{1+\eta-u}{2}} \quad \text{for } |\mathfrak{Im}(w)| \geq 1000. \tag{4.47}$$

From the definition (2.4), we have the trivial bound

$$|M_X(w)| \leq X^{1+\eta}. \tag{4.48}$$

It follows from

$$|h_X(w)| \leq |\zeta(w)M_X(w)|^2 + 2|\zeta(w)||M_X(w)|,$$

the bounds (4.47), (4.48), $\frac{|w+1|}{2\pi} > 1, -\frac{1+\eta-\mathfrak{Re} w}{2} < 0$, and $X \geq kH_0$, that

$$|h_X(w)| \leq b_7(k,\eta,H_0)\Big(\frac{|w+1|}{2\pi}\Big)^{1+\eta-u} X^{2(1+\eta)} \quad \text{for } |\mathfrak{Im}(w)| \geq 1000, \tag{4.49}$$

with

$$b_7(k,\eta,H_0) = \left(1 + \frac{2}{3.006\zeta(1+\eta)(kH_0)^{1+\eta}}\right)(3.006\zeta(1+\eta))^2. \tag{4.50}$$

We apply this with $w = 1+\eta+(1+2\eta)e^{i\theta} \pm iU$. Since $\cos\theta \le 0$, a little calculation gives

$$|w+1| = |2+\eta+(1+2\eta)e^{i\theta} \pm iU| \le \sqrt{(2+\eta)^2+(1+2\eta+U)^2} \le b_8(\eta,H)U,$$

with

$$b_8(\eta,H) = \sqrt{\frac{(2+\eta)^2}{H^2} + \Big(\frac{1+2\eta}{H}+1\Big)^2}. \tag{4.51}$$

In addition $1+\eta-u = 1+\eta-(1+\eta+(1+2\eta)\cos\theta) = -(1+2\eta)(\cos\theta)$, and (4.49) gives

$$|h_X(1+\eta+(1+2\eta)e^{i\theta} \pm iU)| \le b_7(k,\eta,H_0)\Big(\frac{b_8(\eta,H)}{2\pi}U\Big)^{-(1+2\eta)(\cos\theta)} X^{2(1+\eta)}, \tag{4.52}$$

since $|\mathfrak{Im}(1+\eta+(1+2\eta)e^{i\theta} \pm iU)| \ge U-(1-2\eta) \ge H-2 \ge 1000$. We use this to bound $\mathscr{L}_k$ as defined in (4.33):

$$\mathscr{L}_k \le \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \Bigg(\log b_7(k,\eta,H_0) - (1+2\eta)(\cos\theta)\log\Big(\frac{b_8(\eta,H)}{2\pi}U\Big) + 2(1+\eta)(\log X)\Bigg)\,d\theta.$$

Calculating the integrals give

$$\mathscr{L}_k \le \frac{\log b_7(k,\eta,H_0)}{2} + \frac{(1+2\eta)}{\pi}\log\Big(\frac{b_8(\eta,H)}{2\pi}U\Big) + (1+\eta)(\log X). \tag{4.53}$$

By (4.43) and (4.46) we may take $M = b_5(\eta)$ and $m = 1-b_6(10^9,\eta)^2$ in (4.30) and (4.31) in the case of $f(s) = h_X(s)$. Therefore by Proposition 4.10

$$|\arg h_X(\sigma+iU)| \le \frac{\pi}{\log 2}\mathscr{L}_k + \frac{\pi\log b_5(\eta)}{2\log 2} - \frac{\pi\log(1-b_6(10^9,\eta)^2)}{\log 2} + \frac{\pi}{2} + o_k(1). \tag{4.54}$$

Inserting the upper bound for $\mathscr{L}_k$ from (4.53) and letting $k\to\infty$ we complete the proof as the $o_k(1)$ terms goes to zero. □

We are now in a position to bound the arguments.

Lemma 4.12. *Let $0 < H \le H_0 \le T$ and $X \le T$. Let $\eta \in (\eta_0, \frac{1}{2})$ with $\eta_0 = 0.23622\ldots$, σ' and μ satisfying $\frac{1}{2} \le \sigma' < 1 < \mu \le 1+\eta$. Then*

$$\Big|\int_{\sigma'}^{\mu} \arg h_X(\tau+iT)d\tau - \int_{\sigma'}^{\mu} \arg h_X(\tau+iH)d\tau\Big| \le C_7(\eta,H)\,(\mu-\sigma')(\log T), \tag{4.55}$$

where

$$C_7(\eta,H) = \frac{2(1+2\eta)+2\pi(1+\eta)}{\log 2} + \frac{b_9(\eta,H)}{\log H_0}. \tag{4.56}$$

with $b_9(\eta,H)$ defined in (4.59).

Proof. Note that

$$\Big|\int_{\sigma'}^{\mu} \arg h_X(\tau+iT)d\tau - \int_{\sigma'}^{\mu} \arg h_X(\tau+iH)d\tau\Big| \le (\mu-\sigma')\max_{\tau\in(\sigma',\mu)}\Big(|\arg h_X(\tau+iT)| + |\arg h_X(\tau+iH)|\Big). \tag{4.57}$$

By Corollary 4.11 we have

$$|\arg h_X(\tau+iH)| + |\arg h_X(\tau+iT)| \le b_9(\eta,H) + \frac{(1+2\eta)}{\log 2}(\log(HT)) + \frac{2\pi(1+\eta)}{\log 2}(\log X) \tag{4.58}$$

with

(4.59)
$$b_9(\eta,H)=\frac{\pi\log b_7(k,\eta,H_0)}{\log 2}+\frac{\pi\log b_5(\eta)}{\log 2}-\frac{2\pi\log(1-b_6(10^9,\eta)^2)}{\log 2}+\pi+\frac{2(1+2\eta)}{\log 2}\log\Big(\frac{b_8(\eta,H)}{2\pi}\Big)$$

where b_7, b_5, b_6, b_8 are defined in (4.50), (4.44), (4.45), (4.51). Factoring $\log T$ in the right hand side of (4.58), using $H\le T$, $X\le T$, and $H_0\le T$ yields

$$|\arg h_X(\tau+iH)|+|\arg h_X(\tau+iT)|\le(\log T)\left(\frac{2(1+2\eta)+2\pi(1+\eta)}{\log 2}+\frac{b_9(\eta,H)}{\log H_0}\right). \tag{4.60}$$

Combining (4.57) and (4.60) leads to (4.55). □

4.5. **Explicit lower bounds for $\int_H^T\log|h_X(\mu+it)|dt$.** First, observe that (4.45) implies for

$$\mu\ge 1+\eta_0=1.23622\ldots,\ \ |f_X(\mu+it)|<1. \tag{4.61}$$

This fact is used in the next lemma.

Lemma 4.13. *Assume $\mu\ge 1+\eta_0$ where $\eta_0=0.23622\ldots$. Let $X=kT$ where $T\ge H_0$, k satisfies* (4.1)*, $k\le 1$, and $2\pi m_0\le H<T$. Then*

$$-\int_H^T\log|h_X(\mu+it)|dt\le C_8(k,\mu)(\log T). \tag{4.62}$$

with

$$C_8(k,\mu)=b_{10}(k,\mu)\frac{(\log(kH_0))^2}{(kH_0)^{2\mu-2}}\left(\frac{4\mu b_{11}(kH_0,2\mu)}{k(2\mu-1)}+\frac{2\pi m_0(2\mu-1)b_{11}(kH_0,2\mu-1)}{(\mu-1)}\right), \tag{4.63}$$

b_{10} is defined in (4.66)*, b_{11} in* (4.68)*, and m_0 in* (3.15)*.*

Proof. We begin by remarking that (4.45) implies $|f_X(\mu+it)|\le b_6(kH_0,\mu-1)<1$ since $X\ge kH_0\ge 10^9$ and $\mu\ge 1+\eta_0$. Next, observe that $|h_X(\mu+it)|\ge|1-f_X(\mu+it)^2|\ge 1-|f_X(\mu+it)|^2$ and thus

$$-\log|h_X(\mu+it)|\le-\log(1-|f_X(\mu+it)|^2). \tag{4.64}$$

Since $-\frac{\log(1-u^2)}{u^2}$ increases with $u\in(0,1)$, we have

$$-\log(1-|f_X(\mu+it)|^2)\le b_{10}(k,\mu)|f_X(\mu+it)|^2, \tag{4.65}$$

with

$$b_{10}(k,\mu)=-\frac{\log\left(1-b_6(kH_0,\mu-1)^2\right)}{b_6(kH_0,\mu-1)^2} \tag{4.66}$$

where b_6 is defined in (4.45). It follows from (4.64) and (4.65) that

$$-\int_H^T\log|h_X(\mu+it)|dt\le b_{10}(k,\mu)\int_H^T|f_X(\mu+it)|^2dt. \tag{4.67}$$

We apply Lemma 3.6 and the bound $|\lambda_X(n)|\le d(n)$ with $\lambda_X(n)=0$ if $n\le X$. We obtain

$$\begin{aligned}\int_H^T|f_X(\mu+it)|^2dt&\le\sum_{n=1}^{\infty}\frac{|\lambda_X(n)|^2}{n^{2\mu}}(T-H+2\pi m_0(n+1))\\&\le(T-H+2\pi m_0)\sum_{n>X}\frac{d(n)^2}{n^{2\mu}}+2\pi m_0\sum_{n>X}\frac{d(n)^2}{n^{2\mu-1}}.\end{aligned}$$

We appeal to (3.13) to bound the above sums:

$$\sum_{n\geq X} \frac{d(n)^2}{n^\tau} \leq \frac{(\log X)^3}{X^{\tau-1}} \frac{2\tau b_{11}(kH_0,\tau)}{(\tau-1)},$$

since $X \geq kH_0$ where

$$b_{11}(X,\tau) = 1 + \frac{3}{(\tau-1)(\log X)} + \frac{6}{(\tau-1)^2(\log X)^2} + \frac{6}{(\tau-1)^3(\log X)^3}. \tag{4.68}$$

Since $X = kT$ we deduce that

$$\int_H^T |f_X(\mu+it)|^2 dt \leq \frac{(\log(kT))^3}{(kT)^{2\mu-2}} \left(\frac{4\mu b_{11}(kH_0,2\mu)}{k(2\mu-1)} + \frac{2\pi m_0(2\mu-1) b_{11}(kH_0,2\mu-1)}{(\mu-1)} \right).$$

Note that $\frac{(\log(kT))^2}{(kT)^{2\mu-2}}$ decreases with T as long as $10^9 > e^{\frac{1}{\mu-1}}$ (i.e. $\mu > \mu_2 = 1.072382\ldots$). Using this and $\log(kT) \leq \log T$ (since $k \leq 1$) implies

(4.69)

$$\int_H^T |f_X(\mu+it)|^2 dt \leq \frac{(\log(kH_0))^2}{(kH_0)^{2\mu-2}} \left(\frac{4\mu b_{11}(kH_0,2\mu)}{k(2\mu-1)} + \frac{2\pi m_0(2\mu-1) b_{11}(kH_0,2\mu-1)}{(\mu-1)} \right) (\log T).$$

We conclude by combining this with (4.67). □

4.6. **Proof of Zero Density Result.** Finally, we are able to compile our bounds to obtain an upper bound for $N(\sigma,T)$.

Lemma 4.14. *Assume* $\alpha > 0, d > 0, \delta > 0, \eta_0 = 0.23622\ldots, \eta \in [\eta_0, \frac{1}{2})$, *and* $\mu \in [1+\eta_0, 1+\eta]$. *Let* $H_0 = 3.0610046 \cdot 10^{10}$, $1002 \leq H \leq H_0$, $\frac{10^9}{H_0} \leq k \leq 1$, $T \geq H_0$, *and* $X = kT$ *Assume* $\sigma > \frac{1}{2} + \frac{d}{\log H_0}$, $\mathcal{U}(\alpha,k,H_0) > 1$, *and* $\mathcal{U}(\alpha,k,T)$ *decreases in* T. *Thus*

$$N(\sigma,T) \leq \frac{(T-H)(\log T)}{2\pi d} \log\left(1 + \mathcal{C}_1 \frac{(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)} T^{\frac{8}{3}(1-\sigma)}}{T-H}\right) + \frac{\mathcal{C}_2}{2\pi d}(\log T)^2, \tag{4.70}$$

$$N(\sigma,T) \leq \frac{\mathcal{C}_1}{2\pi d}(\log(kT))^{2\sigma}(\log T)^{5-4\sigma} T^{\frac{8}{3}(1-\sigma)} + \frac{\mathcal{C}_2}{2\pi d}(\log T)^2, \tag{4.71}$$

with

(4.72)

$$\mathcal{C}_1 = \mathcal{C}_1(\alpha,d,\delta,k,H,\sigma) = b_{12}(H) e^{\frac{8}{3}\delta(2\sigma-1)M(k,\delta) + \frac{4\delta(2\sigma-1)\log\log H_0}{\log(kH_0)+2\delta}} \mathcal{U}(\alpha,k,H_0)^{2(1-\sigma)+\frac{2d}{\log H_0}+\frac{2\delta(2\sigma-1)}{\log(kH_0)+2\delta}} \times$$
$$\mathcal{V}(\alpha,k,\delta,H_0)^{2\sigma-1} e^{\frac{2d(2\log\log H_0 - \log\log(kH_0))}{\log H_0} + \frac{8d}{3} + 2\alpha},$$

(4.73)

$$\mathcal{C}_2 = \mathcal{C}_2(d,\eta,k,H,\mu,\sigma) = C_7(\eta,H)\left(\mu - \sigma + \frac{d}{\log H_0}\right) + C_8(k,\mu),$$

and $\mathcal{U}, \mathcal{V}, M(k,\delta), C_7, C_8$ *and* b_{12} *are respectively defined in* (4.18), (4.19), (4.23), (4.56), (4.63), *and* (4.75).

Remark. 1. The assumptions that $U(\alpha,k,H_0) > 1$ and $U(\alpha,k,T)$ are decreasing can be removed from the theorem. However, this would overly complicate the statement of the theorem. In all instances that we apply this theorem (for various values of α and k) these conditions hold.

Proof. We begin by assuming that T is not the ordinate of a zero of $\zeta(s)$. From (2.3), (2.9), and the definition (2.10) of F_X, we have for $\sigma \in [\sigma', 1]$ where $\sigma' \geq \frac{1}{2}$ and $\mu \in [1+\eta_0, 1+\eta]$

$$N(\sigma,T) \leq \frac{1}{2\pi(\sigma-\sigma')}\Big((T-H)\log\left(1+\frac{F_X(\sigma',T)-F_X(\sigma',H)}{(T-H)}\right) + \int_{\sigma'}^{\mu} \arg h_X(\tau+iT)d\tau - \int_{\sigma'}^{\mu} \arg h_X(\tau+iH)d\tau - \int_H^T \log|h_X(\mu+it)|dt\Big).$$

We apply Lemma 4.9, Lemma 4.12, and Lemma 4.13 to achieve

(4.74)

$$N(\sigma,T) \leq \frac{(T-H)}{2\pi(\sigma-\sigma')}\times \log\left(1+\frac{e^{\frac{8}{3}\delta(2\sigma'-1)M(k,\delta)+\frac{4\delta(2\sigma'-1)\log\log H_0}{\log(kH_0)+2\delta}}\,\mathcal{U}(\alpha,k,T)^{2(1-\sigma')+\frac{2\delta(2\sigma'-1)}{\log(kT)+2\delta}}\,\mathcal{V}(\alpha,k,\delta,T)^{2\sigma'-1-\frac{2\delta(2\sigma'-1)}{\log(kT)+2\delta}}}{2(\omega_2(\sigma',T,\alpha))^2}\times \frac{(\log(kT))^{2\sigma'}(\log T)^{4(1-\sigma')}T^{\frac{8}{3}(1-\sigma')}}{(T-H)}\right)+\frac{(C_7(\eta,H)\,(\mu-\sigma')+C_8(k,\mu))\,(\log T)}{2\pi(\sigma-\sigma')}.$$

We make the choice $\sigma' = \sigma - \frac{d}{\log T}$, for some $d > 0$. From the definition (4.19), we note that $\mathcal{V}(\alpha,k,\delta,T)$ decreases with T. Since by assumption $\mathcal{U}(\alpha,k,H_0) > 1$ and $T \to U(k,\alpha,T)$ decreases, it follows that $\mathcal{U}(\alpha,k,H_0)^{\frac{2d}{\log T}+\frac{2\delta(2\sigma'-1)}{\log(kT)+2\delta}}$ decreases with T and thus

$$\mathcal{U}(\alpha,k,T)^{2(1-\sigma')+\frac{2\delta(2\sigma'-1)}{\log(kT)+2\delta}} \leq \mathcal{U}(\alpha,k,H_0)^{2(1-\sigma)+\frac{2d}{\log H_0}+\frac{2\delta(2\sigma'-1)}{\log(kH_0)+2\delta}}.$$

It may be shown that for our choice of parameters α, k, δ that $\mathcal{V}(\alpha,k,\delta,T) > 1$ for all $T \geq H_0$ and thus

$$\mathcal{V}(\alpha,k,\delta,T)^{2\sigma'-1-\frac{2\delta(2\sigma'-1)}{\log(kT)+2\delta}} \leq \mathcal{V}(\alpha,k,\delta,T)^{2\sigma'-1}.$$

In addition,

$$\begin{aligned}(\log(kT))^{2\sigma'}(\log T)^{4(1-\sigma')}T^{\frac{8}{3}(1-\sigma')} &= e^{\frac{2d}{\log T}(2\log\log T-\log\log(kT))+\frac{8d}{3}}(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)}T^{\frac{8}{3}(1-\sigma)}\\ &\leq e^{\frac{2d(2\log\log H_0-\log\log(kH_0))}{\log H_0}+\frac{8d}{3}}(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)}T^{\frac{8}{3}(1-\sigma)},\end{aligned}$$

since $T \geq H_0$ and $\frac{10^9}{H_0} \leq k \leq 1$ imply $\frac{2\log\log T-\log\log(kT)}{\log T}$ decreases in T. Since $\omega_2(\sigma',T,\alpha)$ as defined in (3.19) increases with $\sigma' \geq \sigma - \frac{d}{\log H_0}$ and decreases with T, then

$$\frac{1}{2(\omega_2(\sigma',T,\alpha))^2} \leq b_{12}(H)e^{2\alpha} \text{ with } b_{12}(H) = \frac{1}{2(1-\frac{1}{H})^2}. \tag{4.75}$$

Combining the above inequalities establishes (4.70), and thus (4.71) (applying $\log(1+y) \leq y$).

(4.76)

$$N(\sigma,T) \leq \frac{(T-H)(\log T)}{2\pi d} \log\Bigg(1 + b_{12}(H) e^{\frac{8}{3}\delta(2\sigma'-1)M(k,\delta) + \frac{4\delta(2\sigma'-1)\log\log H_0}{\log(kH_0)+2\delta}}$$
$$\times\, \mathcal{U}(\alpha,k,H_0)^{2(1-\sigma)+\frac{2d}{\log H_0}+\frac{2\delta(2\sigma'-1)}{\log(kH_0)+2\delta}} \mathcal{V}(\alpha,k,\delta,H_0)^{2\sigma-1} e^{\frac{2d(2\log\log H_0 - \log\log(kH_0))}{\log H_0}+\frac{8d}{3}+2\alpha}$$
$$\times\, \frac{(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)} T^{\frac{8}{3}(1-\sigma)}}{(T-H)}\Bigg)$$
$$+\, \frac{\Big(C_7(\eta,H)\left(\mu-\sigma+\frac{d}{\log H_0}\right)+C_8(k,\mu)\Big)(\log T)^2}{2\pi d}.$$

Since $\sigma' \leq \sigma$, each remaining occurrence of σ' may be replaced by σ. Finally, by a continuity argument these inequalities extend to the case where T is the ordinate of a zero of the zeta function. □

5. Tables of Computation

For fixed values of σ, Table 1 provides bounds for $N(\sigma,T)$ of the shape (4.71). We fix values for k in $[\frac{10^9}{H_0}, 1]$. The parameters α, d, δ, η and H are chosen to make $\frac{\mathcal{C}_1}{2\pi d}$ as small as possible with $\mathcal{C}_1(\alpha,d,\delta,k,H,\sigma)$ as defined in (4.72) . The program returns $H = H_0 - 1$ for all lines in the table. With this H we minimize of $C_7(\eta,H)$ which chooses $\eta = 0.25618\ldots$. Then μ is chosen to minimize $\mu C_7(\eta,H)+C_8(k,\mu)$ (as in the definition (4.73) of $\mathcal{C}_2 = \mathcal{C}_2(d,\eta,k,H,\mu,\sigma)$). We remark that there is a small bit of subtlety when considering $\mathcal{U}(\alpha,k,T)$, it is necessary to ensure all the coefficients in $\mathcal{J}(k,T)$ are positive and this is checked with each set of parameters used. This is to guarantee that $\mathcal{U}(\alpha,k,T)$ decreases with T.

For fixed values of σ, Table 2 provide bounds for $N(\sigma,H_0)$ of the shape (4.70). In this case, the choice of H is essential and we choose $H = H_0 - 10^{-6}$. As a consequence the "main term" is $\frac{10^{-6}}{2\pi d}(\log H_0)\log\left(1+10^6\mathcal{C}_1(\log(kH_0))^{2\sigma}(\log H_0)^{4(1-\sigma)}H_0^{\frac{8}{3}(1-\sigma)}\right)$ which becomes insignificant in comparison to $\frac{\mathcal{C}_2(d,\eta,k,H,\mu,\sigma)}{2\pi d}(\log H_0)^2$, the term arising from the argument. We take $\alpha = 0.324$, $\delta = 0.3000$, and $k = 1$ (as we did not find any other values giving better bounds). The parameter η is chosen to minimize $C_7(\eta,H)$, and then μ to minimize $\mu C_7(\eta,H)+C_8(k,\mu)$: $\eta = 0.2561\ldots$ and $\mu = 1.2453\ldots$.

TABLE 1. The bound $N(\sigma,T) \leq A(\log(kT))^{2\sigma}(\log T)^{5-4\sigma}T^{\frac{8}{3}(1-\sigma)} + B(\log T)^2$ (4.71) for $\sigma \geq \sigma_0$ with $\frac{10^9}{H_0} \leq k \leq 1$.

σ_0	k	μ	α	δ	d	$A = \frac{\mathcal{C}_1}{2\pi d}$	$B = \frac{\mathcal{C}_2}{2\pi d}$
0.60	0.5	1.251	0.288	0.3140	0.341	2.177	5.663
0.65	0.6	1.249	0.256	0.3070	0.340	2.963	5.249
0.70	0.8	1.247	0.222	0.3040	0.339	3.983	4.824
0.75	1.0	1.245	0.189	0.3030	0.338	5.277	4.403
0.80	1.0	1.245	0.160	0.3030	0.337	6.918	3.997
0.85	1.0	1.245	0.133	0.3030	0.336	8.975	3.588
0.86	1.0	1.245	0.127	0.3030	0.335	9.441	3.514
0.87	1.0	1.245	0.122	0.3030	0.335	9.926	3.430
0.88	1.0	1.245	0.116	0.3030	0.335	10.431	3.346
0.89	1.0	1.245	0.111	0.3030	0.335	10.955	3.262
0.90	1.0	1.245	0.105	0.3030	0.334	11.499	3.186
0.91	1.0	1.245	0.100	0.3030	0.334	12.063	3.102
0.92	1.0	1.245	0.095	0.3030	0.334	12.646	3.017
0.93	1.0	1.245	0.089	0.3030	0.333	13.250	2.941
0.94	1.0	1.245	0.084	0.3030	0.333	13.872	2.856
0.95	1.0	1.245	0.079	0.3030	0.333	14.513	2.772
0.96	1.0	1.245	0.074	0.3030	0.332	15.173	2.694
0.97	1.0	1.245	0.069	0.3030	0.332	15.850	2.609
0.98	1.0	1.245	0.064	0.3030	0.331	16.544	2.532
0.99	1.0	1.245	0.060	0.3030	0.331	17.253	2.446

TABLE 2. Bound (4.70) with $k = 1$

σ	d	$\frac{1}{2\pi d}$	$\mathcal{C}_1$	$\frac{\mathcal{C}_2}{2\pi d}$	$N(\sigma, H_0) \leq$
0.60	2.414	0.066	2094.73	0.893	520.28
0.65	3.621	0.044	97986.60	0.595	346.85
0.70	4.828	0.033	4583580.34	0.447	260.14
0.75	6.036	0.027	214409007.32	0.357	208.11
0.80	7.243	0.022	10029544375.44	0.298	173.42
0.85	8.450	0.019	469158276689.92	0.255	148.65
0.86	8.691	0.019	1012341447042.27	0.248	144.52
0.87	8.933	0.018	2184412502812.95	0.242	140.61
0.88	9.174	0.018	4713486735514.76	0.235	136.91
0.89	9.416	0.017	10170678467214.40	0.229	133.40
0.90	9.657	0.017	21946110446020.33	0.224	130.07
0.91	9.899	0.017	47354929689448.17	0.218	126.90
0.92	10.140	0.016	102181631292174.11	0.213	123.88
0.93	10.382	0.016	220485720114084.42	0.208	120.99
0.94	10.623	0.015	475760194464125.94	0.203	118.24
0.95	10.864	0.015	1026586948666903.92	0.199	115.62
0.96	11.106	0.015	2215151194732183.30	0.195	113.10
0.97	11.347	0.015	4779814142285142.58	0.190	110.70
0.98	11.589	0.014	10313798574616601.14	0.186	108.39
0.99	11.830	0.014	22254932487167323.15	0.183	106.18

REFERENCES

[1] R.J. Backlund, *Über die Nullstellen der Riemannschen Zetafunktion*, Acta Mathematica, vol. 41 (1917), 345–375.

[2] H. Bohr, E. Landau, *Beiträge zur Theorie der Riemannschen Zetafunktion*, Math. Ann. 74 (1913), no. 1, 3–30.
[3] J. Bourgain, *On large values estimates for Dirichlet polynomials and the density hypothesis for the Riemann zeta function*, Internat. Math. Res. Notices 2000, no. 3, 133-146.
[4] P. Dusart, *Explicit estimates of some functions over primes*, Ramanujan J, (2016), doi:10.1007/s11139-016-9839-4.
[5] L. Faber, H. Kadiri, *New bounds for $\psi(x)$*, Math. Comp. 84 (2015), no. 293, 1339–1357.
[6] T. Gowers, *Vinogradovs three primes theorem*, notes available at `https://www.dpmms.cam.ac.uk/ wtg10/` .
[7] G. H. Hardy, A. E. Ingham, G. Pólya, *Theorems concerning mean values of analytic functions*, Proceedings Royal Soc. London (A) 113 (1927), 542-569.
[8] G.H. Hardy, J.E. Littlewood, *The zeros of the Riemann zeta-function on the critical line*, Math. Z. 10 (1921), 283–317.
[9] G. A. Hiary, *An explicit van der Corput estimate for $\zeta(\frac{1}{2}+it)$*, Indag. Math. (N.S.) (2016), no. 2, 524-533.
[10] M.N. Huxley, *On the difference between consecutive primes*, Invent. Math. 15 (1972), 164-170.
[11] M.N. Huxley, *Large values of Dirichlet polynomials. II*, Collection of articles in memory of Juriĭ Vladimirovič Linnik, Acta Arith. 27 (1975), 159-169.
[12] A. E. Ingham *On the difference between consecutive primes*, Quart. J. Pure and Appl. Math., Oxford, (2) 8 (1937), 255–266.
[13] M. Jutila, *Zero-density estimates for L-functions*, Acta Arith. 32 (1977), 52-62.
[14] H. Kadiri *A zero density result for the Riemann zeta function*, Acta Arith. 160 (2013), no. 2, 185–200.
[15] H. Kadiri, A. Lumley *Short effective intervals containing primes*, Integers 14 (2014), Paper No. A61, 18 pp.
[16] H. Kadiri, N. Ng, *Explicit zero density theorems for Dedekind zeta functions*, J. Number Theory 132 (2012), no. 4, 748-775.
[17] K.S. McCurley, *Explicit estimates for the error term in the prime number theorem for arithmetic progressions*, Math. Comp. 42 (165) (1984) 265-285.
[18] H.L. Montgomery, R.C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. (2) 8 (1974), 73-82.
[19] D.J. Platt, *Isolating some non-trivial zeros of zeta*, Math. Comp. 86 (2017), no. 307, 2449–2467.
[20] D.J. Platt, T.S. Trudgian, *On the first sign change of $\theta(x)-x$*, Math. Comp. 85 (2016), no. 299, 1539–1547.
[21] E. Preissmann, *Sur une inégalité de Montgomery-Vaughan*, Enseign. Math. (2) 30 (1984), no. 1-2, 95113.
[22] H. Rademacher, *On the Phragmén-Lindelöf theorem and some applications*, Math. Z. 72 (1959), 192–204.
[23] O. Ramaré, *On Snirelman's constant*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 22 (1995), no. 4, 645–706.
[24] O. Ramaré, *An explicit density estimate for Dirichlet L-functions*, Math. Comp. 85 (2016), 325-356.
[25] E.C. Titchmarsh, *The Theory of the Riemann Zeta-function*, second edition, Oxford Science Publications.
[26] T. Trudgian, *Updating the error term in the prime number theorem*, Ramanujan J. 39 (2016), no. 2, 225234.

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