NEW BOUNDS FOR $\psi(x)$

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ABSTRACT. In this article we provide new explicit Chebyshev's bounds for the prime counting function $\psi(x)$. The proof relies on two new arguments: smoothing the prime counting function which allows to generalize the previous approaches, and a new explicit zero density estimate for the zeros of the Riemann zeta function.

1. INTRODUCTION.

1.1. **Main Theorem and History.** We recall that $\psi(x)$ is the Chebyshev function given by

$$\psi(x) = \sum_{n\le x} \Lambda(n), \text{ with } \Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for } k \ge 1, \\ 0 & \text{else.} \end{cases}$$

The Prime Number Theorem (PNT) is equivalent to

$$\psi(x) \sim x \text{ as } x \to \infty.$$

This estimate is a core tool in solving many problems in number theory and an explicit form of it turns out to be very useful in a wide range of problems. In this article, we investigate explicit bounds (also known as Chebyshev's bounds) for the error term

$$E(x) = \left| \frac{\psi(x) - x}{x} \right|.$$

For instance, the main article of reference [20] in this subject is extensively used in various fields including Diophantine approximation, cryptography, and computer science. Moreover, breakthroughs concerning Goldbach's conjecture (see the work of Ramaré [18], Tao [25], and Helfgott [6] [7]) rely on sharp explicit bounds for finite sums over primes. We combine a new explicit zero density estimate for $\zeta(s)$ and an optimized smoothing argument to prove

Theorem 1.1. *Let $b_0 \le 9963$ be a fixed positive constant. Let $x \ge e^{b_0}$. Then there exists $\epsilon_0 > 0$ such that $E(x) \le \epsilon_0$, where ϵ_0 is given explicitly in* (3.9) *and is computed in Table 3.*

Corollary 1.2. *For all $x \ge e^{20}$, $E(x) \le 5.3688 \cdot 10^{-4}$.*

A classic explicit formula that relates prime numbers to non-trivial zeros of ζ is given by [1, §17, (1)]:

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1 - x^{-2}), \tag{1.1}$$

when x is not a prime power. As the sum over the zeros is not absolutely convergent, it is impossible to directly use this formula to bound the error term $E(x)$. To bypass this problem, the standard argument is to apply an explicit formula to an average of $\psi(x)$ on a small interval containing $[0, x]$.

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In 1941 Rosser [22, Theorem 12] provides an explicit version of this proof. In 1962 Rosser and Schoenfeld [23, Theorem 28] improve on this method by introducing further averaging. Later results of Rosser and Schoenfeld [24], Dusart [2] [3], and very recently Nazardonyavi and Yakubovich [14] all use the argument of [23]. They successively obtain smaller bounds for the error term as a consequence of improvements concerning the location of the non-trivial zeros of the Riemann zeta function, namely the verification of the Riemann Hypothesis up to a fixed height H, and an explicit zero-free region of the form $\mathfrak{Re} s \geq 1 - \dfrac{1}{R \log |\mathfrak{Im} s|}$ and $|\mathfrak{Im} s| \geq 2$, where R is a computable constant. On the other hand Theorem 1.1 relies on new arguments. We introduce a smooth weight f and compare $\psi(x)$ to the sum $\mathscr{S}(x) = \sum_{n\geq 1} \Lambda(n) f\left(\frac{n}{x}\right)$. In Section 3.1 we choose f in a close to optimal way so as to make the bound on $E(x)$ as small as possible. We also observe that Rosser and Schoenfeld's averaging method is a special case of this smoothing method (see Section 3.4 for further discussion). In Theorem 2.3 we establish a general explicit formula for $\mathscr{S}(x)$. A large contribution to the size of $E(x)$ arises from a sum over the non-trivial zeros of the form $\sum_\rho x^{\rho-1} F(\rho)$, where F is the Mellin transform of f. This sum is studied in Section 2.3. We split it so as to isolate zeros closer to the 1-line (say of real part larger than a fixed σ_0) as they contribute the most to the sum. In section 2.3.2 we estimate this contribution by using for the first time explicit estimates for the zero density $N(\sigma_0, T)$ (as given in article [9]). This allows an extra saving over previous methods as they are of size between $\log T$ and T smaller than $N(T)$. Finally Theorem 2.8 provides a general form for the bound of the error term $E(x)$.

We provide here a history of numerical improvements for Theorem 1.1 in the case where $b_0 = 50$. At the same time we mention which height H and constant R were used.

TABLE 1. For all $x \geq e^{50}$, $E(x) \leq \epsilon_0$.

Authors	H		R		ϵ_0
Rosser [22]	1 467	[22]	17.72	[22]	$1.1900 \cdot 10^{-2}$
Rosser and Schoenfeld [23]	21 943	[12] [13]	17.5163...	[23]	$1.7202 \cdot 10^{-3}$
Rosser and Schoenfeld [24]	1 894 438	[24]	9.645908801	[24]	$1.7583 \cdot 10^{-5}$
Dusart [2]	545 439 823	[26]	9.645908801	[24]	$9.0500 \cdot 10^{-8}$
Dusart [3]*	2 445 999 556 030	[5]*	5.69693	[8]	$1.3010 \cdot 10^{-9}$
Nazardonyavi and Yakubovich [14]*	2 445 999 556 030	[5]*	5.69693	[8]	$1.3055 \cdot 10^{-9}$
Faber and Kadiri	2 445 999 556 030	[5]*	5.69693	[8]	$9.4602 \cdot 10^{-10}$
	30 610 046 000	[17] [16]	5.69693	[8]	$2.3643 \cdot 10^{-9}$

(* unpublished)

Note that when we use the same values for H and R than [3] and [14], our bounds for $E(x)$ are consistently smaller than theirs (for all b_0 except for $b_0 = 10\,000$ in the case of [3]).

1.2. **Zeros of the Riemann zeta function.** We use the latest computations of Platt [16] [17] concerning the verification of RH:

Theorem 1.3. *Let* $H = 3.061 \cdot 10^{10}$. *If* $\zeta(s) = 0$ *at* $0 \leq \mathfrak{Re}(s) \leq 1$ *and* $0 \leq \mathfrak{Im}(s) \leq H$, *then* $\mathfrak{Re}(s) = \frac{1}{2}$.

Table 3 presents values of ϵ_0 computed for this value of H. Prior to the work of Platt, Gourdon [5] announced a verification up to $H = 2\,445\,999\,556\,030$. We choose to use Platt's value of H since his verification of RH is the most rigorous to date (he employs interval arithmetic). Since other recent results ([3] and [14]) use Gourdon's H, we also give a version of Theorem 1.1 based on his value (see Table 4).

From [8, Theorem 1.1] we have the zero-free region:

Theorem 1.4. *Let* $R = 5.69693$. *Then there are no zeros of* $\zeta(s)$ *in the region*

$$\mathfrak{Re} s \geq 1 - \frac{1}{R \log |\mathfrak{Im} s|} \text{ and } |\mathfrak{Im} s| \geq 2.$$

Let $T \geq 2$ and $N(T)$ be the number of non-trivial zeros $\varrho = \beta + i\gamma$ in the region $0 \leq \gamma \leq T$ and $0 \leq \beta \leq 1$. In 1941, Rosser [22, Theorem 19] proved

Theorem 1.5. *Let $T \geq 2$,*

$$P(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8},\ R(T) = a_1 \log T + a_2 \log\log T + a_3,$$

and $a_1 = 0.137$, $a_2 = 0.443$, $a_3 = 1.588$. Then

$$|N(T) - P(T)| \leq R(T).$$

We recall that $N(\sigma_0, T)$ is the number of non-trivial zeros in the region $\sigma_0 \leq \mathfrak{Res} \leq 1$ and $0 \leq \mathfrak{Ims} \leq T$. In [9] the second author proved explicit upper bounds for $N(\sigma_0, T)$:

Theorem 1.6. *Let $3/5 \leq \sigma_0 < 1$. Then there exists constants c_1, c_2, c_3 such that, for all $T \geq H$,*

$$N(\sigma_0, T) \leq c_1 T + c_2 \log T + c_3.$$

The c_i's depend on various (hidden) parameters and it is possible to choose these so as to make the above bound smaller when T is asymptotically large or when it is close to H, the height of the numerical verification of RH. Table 2 at the end of this paper list values for the c_i's in these respective cases. For instance, it gives

$$N(89/100, T) \leq 0.4617T + 0.6644 \log T - 340\,272,$$

which provides a saving of about $1/3(\log T)$ compared to Theorem 1.5.

When T is near H, Theorem 1.6 yields values for the c_i's which provide a bound for $N(\sigma, T)$ of size about $\log H$. For instance, it gives that $N(99/100, H) \leq 78$ while Rosser's Theorem gives $5.2 \cdot 10^{10}$.

2. General form of an explicit inequality for $\psi(x)$.

2.1. Introducing a smooth weight f.

Definition 2.1. *Let $0 < a < b, m \in \mathbb{N}$ and $m \geq 2$. We define a function f on $[a, b]$ by $f(x) = 1$ if $0 \leq x \leq a$, $f(x) = 0$ if $x \geq b$, and $f(x) = g\left(\frac{x-a}{b-a}\right)$ if $a \leq x \leq b$, where g is a function defined on $[0, 1]$ satisfying*

Condition 1: $0 \leq g(x) \leq 1$ *for* $0 \leq x \leq 1$,

Condition 2: *g is an m-times differentiable function on $(0, 1)$ such that for all $k = 1, \dots, m$,*

$$g^{(k)}(0) = g^{(k)}(1) = 0,$$

and there exist positive constants a_k such that

$$|g^{(k)}(x)| \leq a_k \text{ for all } 0 < x < 1.$$

We now consider

$$\mathscr{S}(x) = \sum_{n=1}^{\infty} \Lambda(n) f\left(\frac{n}{x}\right) \text{ and } E_{\mathscr{S}}(x) = \left|\frac{\mathscr{S}(x) - x}{x}\right|. \tag{2.1}$$

Let $\delta > 0$. We denote f^- and f^+ for the function f defined above with the choices $a = 1 - \delta, b = 1$ and $a = 1, b = 1 + \delta$ respectively. We also define $\mathscr{S}^-$ and $\mathscr{S}^+$ the sums $\mathscr{S}$ associated to f^- and f^+ respectively. Observe that

$$\mathscr{S}^-(x) \leq \psi(x) \leq \mathscr{S}^+(x) \text{ and } E(x) \leq \max\left(E_{\mathscr{S}^-}(x), E_{\mathscr{S}^+}(x)\right). \tag{2.2}$$

The Mellin Transform of f is given by

$$F(s) = \int_0^{\infty} f(t) t^{s-1} dt. \tag{2.3}$$

We recall the property (see [10, page 80, (3.1.3)]): if there exist α and β such that $\alpha < \beta$ and, for every $\epsilon > 0$, $f(x) = \mathcal{O}(x^{-\alpha-\epsilon})$ as $x \to 0$, and $f(x) = \mathcal{O}(x^{-\beta+\epsilon})$ as $x \to +\infty$, then F is analytic in $\alpha < \mathfrak{Res} < \beta$. It follows from our choice of f that F is analytic in $\mathfrak{Res} > 0$. Moreover, we have the inverse Mellin transform formula

$$f(t) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s) t^{-s} ds. \tag{2.4}$$

Observe that

$$\int_a^b |f^{(m+1)}(t)| t^{m+1} dt = \frac{1}{(b-a)^m} \int_0^1 |g^{(m+1)}(u)| \left((b-a)u + a\right)^{m+1} du.$$

Let k be a non-negative integer. We define

$$M(a,b,k) = \int_0^1 |g^{(k+1)}(u)| \left((b-a)u + a\right)^{k+1} du. \tag{2.5}$$

We now record some properties of F.

Lemma 2.2. *Let $0 < a < b, m \in \mathbb{N}, m \geq 2$. Let f and g be functions as in Definition 2.1.*

(a) *The Mellin transform F of f has a single pole at $s = 0$ with residue 1 and is analytic everywhere else.*
(b) *Let $s \in \mathbb{C}$ such that $\mathfrak{Res} \leq 1$. Then F satisfies*

$$F(1) = a + (b-a) \int_0^1 g(u) du, \tag{2.6}$$

$$|F(s)| \leq \frac{M(a,b,k)}{(b-a)^k |s|^{k+1}}, \quad \text{for all } \; k = 0, \ldots, m. \tag{2.7}$$

Proof. The identity (2.6) follows immediately from the definition of f.
We now use Condition 1 and Condition 2. We have $F(s) = \int_0^b f(t) t^{s-1} dt$ with $f'(x) = 0$ for $0 < x < a$. We integrate by parts once and observe that $F(s) = \frac{G(s)}{s}$, where

$$G(s) = -\int_a^b f'(t) t^s dt \tag{2.8}$$

is an entire function. The residue of F at $s = 0$ is $G(0) = 1$.
Let $\mathfrak{Res} \leq 1$ and $k = 0, \ldots,$ or m. Inequality (2.7) is obtained by integrating F by parts $k+1$ times:

$$F(s) = \frac{(-1)^{k+1}}{s(s+1)\ldots(s+k)} \int_a^b f^{(k+1)}(t) t^{s+k} dt. \tag{2.9}$$

We consider

$$G_m(s) = \int_a^b t^{s+m} f^{(m+1)}(t) dt.$$

Since $f^{(i)}$ vanishes at both a and b for all $i = k, \ldots, m$, we have

$$G_m(-k) = (m-k)!(-1)^{m-k} \int_a^b f^{(k+1)}(t) dt = (m-k)!(-1)^{m-k} (f^{(k)}(b) - f^{(k)}(a)) = 0. \tag{2.10}$$

Thus F only has a pole at $s = 0$ and is analytic everywhere else. □

2.2. **An explicit formula for a smooth form of $\psi(x)$.** We use classical techniques to rewrite $\mathscr{S}(x)$ as a complex integral, shift the integration contour to the left, and collect all the poles of the integrand so as to obtain a smooth analogue of the classical explicit formula (1.1).

Theorem 2.3. *Let $0 < a < b, m \in \mathbb{N}, m \geq 2$. Let f be a function satisfying Definition 2.1 and F its Mellin transform. Then*

$$\mathscr{S}(x) = xF(1) - \sum_{\rho} x^{\rho} F(\rho) - \frac{\zeta'}{\zeta}(0) - \sum_{n=1}^{\infty} x^{-2n} F(-2n),$$

where ρ runs through all the non-trivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta function.

Proof. We insert (2.4) in (2.1):

$$\mathscr{S}(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} x^s F(s)\Big(- \frac{\zeta'}{\zeta}(s)\Big) ds.$$

Fix $k \in \mathbb{R}\backslash 2\mathbb{N}$ and $T \geq 2$ such that T does not equal an ordinate of a zero of ζ. Observe that the integrand has a pole at $s = 0$ with residue $-\frac{\zeta'}{\zeta}(0)$, a pole at $s = 1$ with residue $xF(1)$, poles at the non-trivial zeros of zeta $\rho = \beta + i\gamma$ with residue $-x^{\rho} F(\rho)$, and poles at the trivial zeros of zeta $s = -2n, n \in \mathbb{N}$, with residue $-x^{-2n} F(-2n)$. We move the vertical line of integration extending from $2 - iT$ to $2 + iT$ to the line of integration extending from $-k - iT$ to $-k + iT$ so as to form the rectangle $\mathscr{R}$. Thus

$$\mathscr{S}(x) = I_1(T, k) + I_2(T, k) - I_3(T, k) - \frac{\zeta'}{\zeta}(0) + F(1)x - \sum_{|\gamma|<T} x^{\rho} F(\rho) - \sum_{1\leq n \leq \frac{k}{2}} x^{-2n} F(-2n),$$

where I_1, I_2, I_3 are respectively integrating along the segments $[-k+iT, 2+iT], [-k+iT, -k-iT], [-k-iT, 2 - iT]$. It remains to prove that for each $j = 1, 2, 3$, $\lim_{k,T\to+\infty} |I_j(T, k)| = 0$. We use the classical bounds (see [1, page 108])

$$\Big|\frac{\zeta'}{\zeta}(\sigma + iT)\Big| \ll \begin{cases} \log^2 T & \text{if } -1 \leq \sigma \leq 2, \\ \log(|\sigma| + T) & \text{if } -k \leq \sigma \leq -1, \end{cases}$$

together with inequality (2.7) for F, and obtain

$$|I_1(T, k)| \ll \frac{\log^2 T}{T^{m+1}} \frac{x^2}{\log x} + \frac{\log T}{T^{m+1}} \frac{1}{x \log x} + \frac{x^{-T}}{T^{m-1}}.$$

We conclude that $\lim_{k,T\to+\infty} |I_1(T, k)| = 0$. Note that $I_3(T, k) = I_1(-T, k)$ converges to 0 by a similar argument. For $I_2(T, k)$, we combine (2.7) with [1, inequality (8)]:

$$|F(-k + it)|\Big|\frac{-\zeta'}{\zeta}(-k + it)\Big| \ll \begin{cases} \frac{\log k}{k^{m+1}} & \text{if } |t| \leq \frac{3}{2}, \\ \frac{\log |t|}{|t|^{m+1}} & \text{if } |t| > \frac{3}{2}. \end{cases}$$

Thus $|I_2(T, k)| \ll x^{-k}\Big(\frac{\log k}{k^{m+1}} + \frac{\log T}{T^m}\Big)$, and $\lim_{k,T\to+\infty} |I_2(T, k)| = 0$. □

2.3. **A general form of explicit bounds for $\psi(x)$.** We deduce from (2.7) that

$$\Big|\sum_{n=1}^{\infty} x^{-2n} F(-2n)\Big| \leq M(a, b, 0) \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} \leq \frac{M(a, b, 0)}{2x^2}.$$

Together with the above, (2.6), and $-\frac{\zeta'}{\zeta}(0) = \frac{\log(2\pi)}{2}$, it follows that

$$E_{\mathscr{S}}(x) \leq \Big|a - 1 + (b - a) \int_0^1 g(u)du\Big| + \sum_{\rho} x^{\beta-1} |F(\rho)| + \log(2\pi)x^{-1} + \frac{M(a, b, 0)}{2} x^{-3}. \tag{2.11}$$

To study the sum over the zeros, we introduce the notation

(2.12)
- $H > 0$ is such that if $\zeta(\beta + i\gamma) = 0$ and $0 < \gamma < H,$ then $\beta = 1/2,$
- $T_0 > 0$ is such that $\sum_{0<\gamma<T_0} \gamma^{-1}$ can be directly computed,
- T_1 is a parameter satisfying $T_0 < T_1 < H,$
- $R \geq 1$ is a constant so that $\zeta(\sigma + it)$ does not vanish in the region $\sigma \geq 1 - \dfrac{1}{R\log|t|}$ and $|t| \geq 2,$
- σ_0 is a parameter satisfying $3/5 \leq \sigma_0 < 1,$
- $c_1 > 0, c_2 > 0, c_3 < 0$ depend on σ_0 so that $N(\sigma_0, T) \leq c_1 T + c_2 \log T + c_3,$ for all $T \geq H.$

Using the symmetry of the zeros of zeta and using the notation $\sum^* = \frac{1}{2}\sum_{\beta=1/2} + \sum_{1/2<\beta<1}$ we have:

$$\sum_{\rho} x^{\beta-1}|F(\rho)| = \sum_{\gamma>0}^{*} \left(x^{\beta-1} + x^{-\beta}\right)\left(|F(\rho)| + |F(\overline{\rho})|\right). \tag{2.13}$$

We now separate the zeros vertically at H:

$$\sum_{\rho} x^{\beta-1}|F(\rho)| = \Sigma_1 + \Sigma_2, \tag{2.14}$$

with

$$\Sigma_1 = x^{-\frac{1}{2}} \sum_{0<\gamma\leq H} \left(|F(1/2 + i\gamma)| + |F(1/2 - i\gamma)|\right),\ \Sigma_2 = \sum_{\gamma>H}^{*} \left(x^{\beta-1} + x^{-\beta}\right)\left(|F(\rho)| + |F(\overline{\rho})|\right).$$

We split Σ_1 vertically at T_1 and use (2.7) to bound $|F(\rho)|$ with $k = 0$ when $\gamma \leq T_1$, and $k = m$ when $T_1 < \gamma \leq H$ respectively. Thus

$$\Sigma_1 \leq 2x^{-\frac{1}{2}}\Big(M(a,b,0)\sum_{0<\gamma\leq T_1}\frac{1}{\gamma} + \frac{M(a,b,m)}{(b-a)^m}\sum_{T_1<\gamma\leq H}\frac{1}{\gamma^{m+1}}\Big). \tag{2.15}$$

Moreover, we split the first sum at height $T_0 \leq T_1$ and denote s_0 a close upper bound for $\sum_{\gamma\leq T_0}\frac{1}{\gamma}$. We use (2.7) with $k = m$ for Σ_2 and split it horizontally at σ_0. Together with the zero-free region given in Theorem 1.4 and the fact that $x^{\beta-1} + x^{-\beta}$ increases with β, we obtain

$$\Sigma_2 \leq 2\frac{M(a,b,m)}{(b-a)^m}\Big(\left(x^{-(1-\sigma_0)} + x^{-\sigma_0}\right)\sum_{\gamma>H}\frac{1}{\gamma^{m+1}} + \sum_{\gamma>H,\sigma_0<\beta<1}\frac{x^{-\frac{1}{R\log\gamma}} + x^{-(1-\frac{1}{R\log H})}}{\gamma^{m+1}}\Big). \tag{2.16}$$

We denote

(2.17)
$$s_1(T_1) = \sum_{0<\gamma\leq T_1}\frac{1}{\gamma},\quad s_2(m,T_1) = \sum_{T_1<\gamma\leq H}\frac{1}{\gamma^{m+1}},\quad s_3(m) = \sum_{\gamma>H}\frac{1}{\gamma^{m+1}},$$
$$s_4(m,\sigma_0) = \sum_{\gamma>H,\sigma_0<\beta<1}\frac{1}{\gamma^{m+1}},\quad s_5(x,m,\sigma_0) = \sum_{\gamma>H,\sigma_0<\beta<1}\frac{x^{-\frac{1}{R\log\gamma}}}{\gamma^{m+1}}.$$

We have

$$\sum_{\rho} x^{\beta-1}|F(\rho)| \le 2\left(M(a,b,0)s_1(T_1) + \frac{M(a,b,m)}{(b-a)^m}s_2(m,T_1)\right)x^{-\frac{1}{2}} + 2\frac{M(a,b,m)}{(b-a)^m}\left(\left(x^{-(1-\sigma_0)} + x^{-\sigma_0}\right)s_3(m) + x^{-(1-\frac{1}{R\log H})}s_4(m,\sigma_0) + s_5(x,m,\sigma_0)\right). \tag{2.18}$$

We conclude by inserting (2.18) in (2.11).

Lemma 2.4. *Let $0 < a < b, m \in \mathbb{N}$, with $m \ge 2$. Let f be a function satisfying Definition 2.1. Let H, T_0, T_1, R, and σ_0 satisfy* (2.12). *Then for all $x > 0$, $E_{\mathscr{S}}(x) \le K(x,a,b,m,\sigma_0)$, where*

$$K(x,a,b,m,\sigma_0) = \left|a - 1 + (b-a)\int_0^1 g(u)du\right| + 2\frac{M(a,b,m)}{(b-a)^m}\Big(\big(x^{-(1-\sigma_0)} + x^{-\sigma_0}\big)s_3(m) + x^{-(1-\frac{1}{R\log H})}s_4(m,\sigma_0) + s_5(x,m,\sigma_0)\Big) + 2\Big(M(a,b,0)s_0 + M(a,b,0)s_1(T_1) + \frac{M(a,b,m)}{(b-a)^m}s_2(m,T_1)\Big)x^{-\frac{1}{2}} + \log(2\pi)x^{-1} + \frac{M(a,b,0)}{2}x^{-3}, \tag{2.19}$$

and $M(a,b,m)$ and the s_i's are defined in (2.5) *and* (2.17) *respectively.*

Note that for a, b, m, σ_0 fixed constants, $K(x,a,b,m,\sigma_0)$ decreases with x. Thus, for all $x \ge x_0$

$$E_{\mathscr{S}}(x) \le K(x_0,a,b,m,\sigma_0). \tag{2.20}$$

2.3.1. *Bounding $s_1(T_1)$, $s_2(m,T_1)$, and $s_3(m)$.* We apply here a result from Rosser and Schoenfeld [24]. It uses explicit estimates for $N(T)$ as given in Theorem 3.4 to bound certain sums over the zeros of zeta.

Lemma 2.5. [24, Lemma 7] *Let $1 < U \le V$, and let $\Phi(y)$ be nonnegative and differentiable for $U < y < V$. Let $(W-y)\Phi'(y) \ge 0$ for $U < y < V$, where W need not lie in $[U,V]$. Let Y be one of U, V, W which is neither greater than both the others or less than both the others. Choose $j = 0$ or 1 so that $(-1)^j(V-W) \ge 0$. Then*

$$\sum_{U<\gamma\le V} \Phi(\gamma) \le \frac{1}{2\pi}\int_U^V \Phi(y)\log\frac{y}{2\pi}dy + (-1)^j\Big(a_1 + \frac{a_2}{\log Y}\Big)\int_U^V \frac{\Phi(y)}{y}dy + E_j(U,V),$$

where the error term $E_j(U,V)$ is given by

$$E_j(U,V) = (1+(-1)^j)R(Y)\Phi(Y) + (N(V)-P(V)-(-1)^jR(V))\Phi(V) - (N(U)-P(U)+R(U))\Phi(U).$$

Corollary 2.6. [24, Corollary of Lemma 7] *If, in addition, $2\pi < U$, then*

$$\sum_{U<\gamma\le V} \Phi(\gamma) \le \Big(\frac{1}{2\pi} + (-1)^j q(Y)\Big)\int_U^V \Phi(y)\log\frac{y}{2\pi}dy + E_j(U,V), \ \ \textit{where}\ \ q(y) = \frac{a_1\log y + a_2}{y\log y\log(y/2\pi)}.$$

Moreover, if $j = 0$ and $W < U$, then

$$E_0(U,V) \le 2R(U)\Phi(U). \tag{2.21}$$

We give details on how we apply Corollary 2.6 and (2.21) to s_1, s_2, and s_3. We take respectively

- $\Phi(y) = y^{-1}, U = T_0, V = T_1,$
- $\Phi(y) = y^{-m-1}, U = T_1, V = H,$
- $\Phi(y) = y^{-m-1}, U = H, V = \infty.$

In each case, $\Phi'(y) \le 0$ for all y, and we choose $W < U$, $Y = U$, and $j = 0$. Since

$$\int_{T_0}^{T_1} \frac{\log \frac{y}{2\pi}}{y} dy = \log(T_1/T_0)\ \log(\sqrt{T_1 T_0}/(2\pi)),$$

$$\int_{U}^{V} \frac{\log \frac{y}{2\pi}}{y^{m+1}} dy = \frac{1 + m \log(U/2\pi)}{m^2 U^m} - \frac{1 + m\log(V/2\pi)}{m^2 V^m},$$

we obtain:

$$s_1(T_1) \le B_1(T_1) = s_0 + \Big(\frac{1}{2\pi} + q(T_0)\Big)\Big(\log(T_1/T_0)\ \log(\sqrt{T_1 T_0}/(2\pi))\Big) + \frac{2R(T_0)}{T_0}, \tag{2.22}$$

$$s_2(m, T_1) \le B_2(m, T_1) = \Big(\frac{1}{2\pi} + q(T_1)\Big)\Big(\frac{1 + m\log(T_1/2\pi)}{m^2 T_1^m} - \frac{1 + m\log(H/2\pi)}{m^2 H^m}\Big) + \frac{2R(T_1)}{T_1^{m+1}}, \tag{2.23}$$

$$s_3(m) \le B_3(m) = \Big(\frac{1}{2\pi} + q(H)\Big)\frac{1 + m\log(H/2\pi)}{m^2 H^m} + \frac{2R(H)}{H^{m+1}}. \tag{2.24}$$

2.3.2. *Bounding $s_4(m, \sigma_0)$ and $s_5(x, m, \sigma_0)$.* We assume here that $\Phi(y) = o(y)$ when $y \to \infty$, so as to ensure that $\lim_{y\to\infty} \Phi(y) N(\sigma_0, y) = 0$. Since all non-trivial zeros of zeta have real part $1/2$ when $\gamma \le H$, then $N(\sigma_0, H) = 0$ and we have the Stieltjes integral

$$\sum_{\gamma \ge H, \beta > \sigma_0} \Phi(\gamma) = -\int_H^\infty N(\sigma_0, y)\Phi'(y) dy.$$

Lemma 2.7. *Let $H, \sigma_0, c_1, c_2, c_3$ satisfy* (2.12)*. Let $H < U \le V$, and let $\Phi(y)$ be non-negative and differentiable for $U < y < V$. Assume $\Phi(y) = o(y)$ when $y \to \infty$ and $(W - y)\Phi'(y) \ge 0$ for all $U < y < V$, where W need not lie in $[U, V]$. Let Y be one of U, V, W which is neither greater than both the others or less than both the others. Then*

$$\sum_{U<\gamma<V, \beta>\sigma_0} \Phi(\gamma) \le (c_1 Y + c_2 \log Y + c_3)\Phi(Y) - (c_1 V + c_2 \log V + c_3)\Phi(V) + \int_Y^V (c_1 + c_2/y)\Phi(y) dy.$$

Proof. We have $0 \le N(\sigma_0, y) \le c_1 y + c_2 \log y + c_3$. Our assumptions ensure us that $\Phi'(y) \ge 0$ if $U \le y \le Y$ and that $\Phi'(y) \le 0$ if $Y \le y \le V$. Thus

$$-\int_U^V N(\sigma_0, y)\Phi'(y) dy \le -\int_Y^V (c_1 y + c_2 \log y + c_3)\Phi'(y) dy,$$

and we integrate by part to complete the proof. □

For $s_4(m, \sigma_0)$, we take $\Phi(y) = \frac{1}{y^{m+1}}$, $\Phi'(y) = -\frac{m+1}{y^{m+2}}$, $W < U = Y = H$, and $V = \infty$. Thus

$$s_4(m, \sigma_0) \le B_4(m, H, \sigma_0) = \left(c_1\big(1 + \frac{1}{m}\big) + c_2 \frac{\log H}{H} + \big(c_3 + \frac{c_2}{m+1}\big)\frac{1}{H}\right)\frac{1}{H^m}. \tag{2.25}$$

For $s_5(x, m, \sigma_0)$, we apply Lemma 2.7 with $U = H$, $V = \infty$, $\Phi(y) = \phi_m(y) = \frac{x^{-\frac{1}{R\log y}}}{y^{m+1}}$, $\phi_m'(y) = \big(\frac{\log x}{R(\log y)^2} - (m+1)\big)\frac{\phi_m(y)}{y}$, and

$$W = e^{\sqrt{\frac{\log x}{R(m+1)}}}. \tag{2.26}$$

Let $J_m(Y)$ denote the integral

$$J_m(Y) = \int_Y^\infty \phi_m(y) dy.$$

We obtain

$$s_5(x, m, \sigma_0) \le (c_1 Y + c_2 \log Y + c_3)\phi_m(Y) + c_1 J_m(Y) + c_2 J_{m+1}(Y), \tag{2.27}$$

Let $z > 0, w \geq 0$. We appeal to the theory of the following modified Bessel function

$$K_\nu(z,w) = \frac{1}{2}\int_w^\infty t^{\nu-1}\exp\Big(-\frac{z}{2}(t+1/t)\Big)\,dt.$$

We do the variable change $y = e^{\frac{z}{2m}t}$, take $z = 2\sqrt{\frac{m\log x}{R}}$, $w = \sqrt{\frac{mR}{\log x}}\log Y = \frac{2m}{z}\log Y$, and recognize

$$J_m(Y) = \frac{z}{2m}K_1(z,w).$$

We use [24, Lemma 4] which asserts that if $w > 1$ then

$$K_1(z,w) \leq Q_1(z,w) = \frac{w^2}{z(w^2-1)}\exp\big(-z/2(w+1/w)\big). \tag{2.28}$$

We deduce for $J_m(Y)$ that if $\log x < mR(\log Y)^2$, then

$$J_m(Y) \leq \frac{R}{2\log x}\frac{(\log Y)^2}{\big(\frac{mR}{\log x}\big)(\log Y)^2 - 1}Y^{-m}e^{-\frac{\log x}{R(\log Y)}}. \tag{2.29}$$

In this case, we have $W < H$, $Y = H$. We insert (2.29) in (2.27) and obtain

$$s_5(x,m,\sigma_0) \leq \big(c_1 + c_2\frac{\log H}{H} + \frac{c_3}{H}\big)\frac{x^{-\frac{1}{R\log H}}}{H^m} + c_1J_m(H) + c_2J_{m+1}(H),$$

We conclude that if $\log x < mR(\log H)^2$ then
(2.30)

$$s_5(x,m,\sigma_0) \leq B_5(x,m,\sigma_0) = \Big(c_1 + c_2\frac{\log H}{H} + \frac{c_3}{H} + \big(c_1 + \frac{c_2}{H}\big)\frac{R}{2\log x}\frac{(\log H)^2}{\big(\frac{mR}{\log x}\big)(\log H)^2 - 1}\Big)\frac{x^{-\frac{1}{R\log H}}}{H^m}.$$

2.3.3. *Main Theorem.* We deduce a new bound for $K(x,a,b,m,\sigma_0)$ from (2.22), (2.23), (2.24), (2.25), and (2.30). Lemma 2.4 becomes

Theorem 2.8. *Let* $0 < a < b, m \in \mathbb{N}$*, with* $m \geq 2$*. Let* f *and* g *be functions satisfying Definition 2.1, and* $M(a,b,m)$ *as defined in* (2.5)*. Let* $H, T_0, T_1, R, \sigma_0, c_1, c_2, c_3$ *satisfy* (2.12)*. Let* x_0 *be a positive constant satisfying* $x_0 < \exp(mR(\log H)^2)$*. Then for all* $x \geq x_0$

(2.31)

$$\begin{aligned}E_{\mathscr{S}}(x) \leq{}& \Big|a - 1 + (b-a)\int_0^1 g(u)du\Big| + \frac{2M(a,b,m)B_5(x_0,m,\sigma_0)}{(b-a)^m} + \frac{2M(a,b,m)B_3(m)}{(b-a)^m}x_0^{-(1-\sigma_0)}\\ &+ \frac{2M(a,b,m)B_3(m)}{(b-a)^m}x_0^{-\sigma_0} + \frac{2M(a,b,m)B_4(m,H,\sigma_0)}{(b-a)^m}x_0^{-(1-\frac{1}{R\log H})}\\ &+ \Big(M(a,b,0)B_1(T_1) + \frac{M(a,b,m)B_2(m,T_1)}{(b-a)^m}\Big)x_0^{-\frac{1}{2}} + \log(2\pi)x_0^{-1} + \frac{M(a,b,0)}{2}x_0^{-3},\end{aligned}$$

where the B_i*'s are defined in* (2.22)*,* (2.23)*,* (2.24)*,* (2.25)*, and* (2.30)*.*

3. NEW EXPLICIT BOUNDS FOR $\psi(x)$.

3.1. **Choosing the smooth function.** We want to find a function g satisfying Definition 2.1 and so that the quotient $\frac{M(a,b,m)}{\int_0^1 g(u)du}$ is as small as possible. By the Cauchy-Schwarz inequality we have

$$M(a,b,m) \leq \sqrt{\frac{b^{2m+3}-a^{2m+3}}{(b-a)(2m+3)}}\sqrt{\int_0^1\big(g^{(m+1)}(u)\big)^2du}. \tag{3.1}$$

It follows from Calculus of Variations (see [4, Chapter 2, §11]) that the function g optimizing the quotient $\frac{\sqrt{\int_0^1 \left(g^{(m+1)}(u)\right)^2 du}}{\int_0^1 g(u)du}$ is given by

$$g(x) = 1 - \frac{(2m+1)!}{(m!)^2} \int_0^x t^m (1-t)^m dt. \tag{3.2}$$

We observe that our choice of kernel is a primitive of the one used in the context of short intervals containing primes by Ramaré & Saouter [21]. This is not surprising as our object of study is $\sum_{n\geq 1} \Lambda(n) f(n/x)$, while theirs is essentially $\sum_{n\geq 1} \Lambda(n) \left(f(n/y) - f(n/x)\right)$.
With definition (3.2), we find

$$\int_0^1 g(u)du = 1 - \frac{(2m+1)!}{(m!)^2} \int_0^1 t^m (1-t)^{m+1} dt = \frac{1}{2}, \tag{3.3}$$

and

$$M(a, b, 0) = \frac{a+b}{2}. \tag{3.4}$$

We use (3.1) to provide a simple bound for $M(a, b, m)$. Since $g(1) = 0, g(0) = 1$, and $g^{(2m+2)}(x) = 0$ for all $0 < x < 1$, integrating by parts m-times leads to

$$\int_0^1 (g^{(m+1)}(u))^2 du = (-1)^m \int_0^1 g^{(2m+1)}(u) \cdot g'(u) du = (-1)^{m+1} g^{(2m+1)}(0) = \frac{(2m)!(2m+1)!}{(m!)^2}.$$

Thus (3.1) becomes

$$M(a, b, m) \leq \lambda(a, b, m) = \sqrt{\frac{b^{2m+3} - a^{2m+3}}{(b-a)(2m+3)}} \cdot \frac{\sqrt{(2m)!(2m+1)!}}{m!}. \tag{3.5}$$

From (3.2), we recognize that

$$g^{(m+1)}(u) = -\frac{(2m+1)!}{m!} P_m(1-2u),$$

where P_m is the m^{th} Legendre polynomial as given by Rodrigues' formula (see [11, formula (0.4)]):

$$P_m(x) = \frac{1}{2^m m!} \frac{\partial^m}{\partial x^m} \left((x^2-1)^m\right).$$

They can be written explicitly (see [11, formula (0.2)]):

$$P_m(x) = \sum_{k=0}^{m} \binom{m}{k}^2 \left(\frac{x+1}{2}\right)^k \left(\frac{x-1}{2}\right)^{m-k}.$$

These polynomials are well-known and are among the built-in functions of PARI/GP. Since the sign of P_m alternates between its roots, $M(a, b, m)$ can be computed directly from

$$M(a, b, m) = \frac{(2m+1)!}{m!} \int_0^1 |P_m(1-2u)| \left((b-a)u + a\right)^{m+1} du. \tag{3.6}$$

3.2. **New explicit bounds for $\psi(x)$.** We rewrite Theorem 2.8 with g as chosen in (3.2):

Theorem 3.1. *Let $m \in \mathbb{N}, m \geq 2$, $\delta > 0$, and the pair (a,b) takes values $(1, 1+\delta)$ or $(1-\delta, 1)$. Let $H, T_0, T_1, R, \sigma_0, c_1, c_2, c_3$ satisfy* (2.12). *Let $b_0 > 0$ be a positive constant satisfying $b_0 < (m+1)R(\log H)^2$. Then for all $x \geq e^{b_0}$*

$$(3.7)\quad E_{\mathscr{S}}(x) \leq \frac{\delta}{2} + \frac{2M(a,b,m)B_5(e^{b_0}, m, \sigma_0)}{\delta^m} + \frac{2M(a,b,m)B_3(m)}{\delta^m} e^{-(1-\sigma_0)b_0} + \frac{2M(a,b,m)B_3(m)}{\delta^m} e^{-\sigma_0 b_0} + \frac{2M(a,b,m)B_4(m,H,\sigma_0)}{\delta^m} e^{-(1-\frac{1}{R\log H})b_0} + \Big(\frac{\delta}{2}B_1(T_1) + \frac{M(a,b,m)B_2(m,T_1)}{\delta^m}\Big) e^{-b_0/2} + \log(2\pi)e^{-b_0} + \frac{M(a,b,0)}{2}e^{-3b_0},$$

where $M(a,b,m)$ is given by (3.6)*, and the B_i's are defined in* (2.22), (2.23), (2.24), (2.25), *and* (2.30).

3.3. **Proof of Theorem 1.1.** We now specify the values for our parameters:we take H and R respectively as in Theorem 1.3 and Theorem 1.4. Let $b_0 \geq 2$ be a fixed constant satisfying $b_0 < 3R(\log H)^2$ (that is $b_0 < 9\,963$ for $H = 3.061 \times 10^{10}$ and $b_0 < 13\,906$ for $H = 2\,445\,999\,556\,030$). Let $x \geq e^{b_0}$. We define

$$(3.8)\quad \epsilon(b_0, a, b, m, \sigma_0, T_1) = \frac{\delta}{2} + \frac{2M(a,b,m)B_5(e^{b_0}, m, \sigma_0)}{\delta^m} + \frac{2M(a,b,m)B_3(m)}{\delta^m} e^{-(1-\sigma_0)b_0} + \frac{2M(a,b,m)B_3(m)}{\delta^m} e^{-\sigma_0 b_0} + \frac{2M(a,b,m)B_4(m,H,\sigma_0)}{\delta^m} e^{-(1-\frac{1}{R\log H})b_0} + \Big(\frac{\delta}{2}B_1(T_1) + \frac{M(a,b,m)B_2(m,T_1)}{\delta^m}\Big) e^{-b_0/2} + \log(2\pi)e^{-b_0} + \frac{M(a,b,0)}{2}e^{-3b_0}.$$

The definition for ϵ_0 follows directly from (2.2) and Theorem 3.1:

$$(3.9)\qquad \epsilon_0 = \max\big(\epsilon(b_0, 1, 1+\delta, m, \sigma_0, T_1), \epsilon(b_0, 1-\delta, 1, m, \sigma_0, T_1)\big).$$

To compute $\epsilon(b_0, 1, 1+d, m, \sigma_0, T_1)$, we choose a value for σ_0 in Table 2, an integer value larger than 2 for m, and a value for δ with up to 4 significant digits. For T_0, we use here a computation of Darcy Best (personal communication) based on Odlyzko's list of zeros [15]: $T_0 = 1\,132\,491$ and $s_0 = 11.637732$. In [24], the authors used $T_0 = 158.84998$ and $s_0 = 0.8113925$. Then we choose a value for T_1 which is either T_0, H or so that it satisfies

$$\frac{\delta}{2}B_1(T_1) = \frac{M(1, 1+\delta, m)B_2(m, T_1)}{\delta^m}.$$

We do the same to compute $\epsilon(b_0, 1-\delta, 1, m, \sigma_0, T_1)$. All values for σ_0, m, and δ are chosen to make ϵ_0 as small as possible.

3.4. **Comparison with Rosser and Schoenfeld's method.**

3.4.1. *The smoothing argument.* The first step of their argument consists in studying $\psi(x)$ on average on a small interval around a large x value. Let $x, \delta > 0$ with $x \notin \mathbb{N}$. Let $m \in \mathbb{N}$. It follows from the First Mean Value Theorem for Integrals applied to $h(z) = \psi(x+z) - (x+z)$ that there exists $z \in (0, \delta x)$ such that:

$$h(z) + z \leq \frac{1}{(\delta/mx)^m} \int_0^{\delta x/m} \cdots \int_0^{\delta x/m} \left(h(y_1 + \ldots + y_m) + (y_1 + \ldots + y_m)\right) dy_1 \ldots dy_m.$$

(In order to make Rosser and Schoenfeld's article consistent with our setup, we replace their δ with our δ/m.) Implementing the explicit formula (1.1) in the right integrals together with the fact that $\psi(x+z) \leq \psi(x)$ leads to [22, Theorems 12 and 14]:

$$(3.10)\qquad E(x) \leq \frac{\delta}{2} + \Sigma(m, \delta, x) + \mathcal{O}(x^{-1}),$$

with

$$\Sigma(m,\delta,x) = \Big|\sum_{\rho} x^{\rho-1} I_{m,\delta}(\rho)\Big|, \text{ and } I_{m,\delta}(\rho) = \frac{\sum_{j=0}^{m}(-1)^{j+m+1}\binom{m}{j}(1+j\delta/m)^{m+\rho}}{(\delta/m)^m \rho(\rho+1)\dots(\rho+m)}.$$

We recall that we obtain (3.10) with

$$\Sigma(m,\delta,x) = \Big|\sum_{\rho} x^{\rho-1} F(\rho)\Big|.$$

We recognize that $I_{m,\delta}$ is indeed the Mellin transform of

$$\nu(t) = \frac{1}{m!}\sum_{j=0}^{m}(-1)^{j+m}\binom{m}{j}\Big(\frac{(1+j\delta/m)-t}{\delta/m}\Big)^m \mathbb{1}\Big(\frac{t}{1+j\delta/m}\Big),$$

where $\mathbb{1}$ is the indicator function on $(0,1)$. Instead we use the function f given by Definition 2.1 and (3.2):

$$f(x) = 1 - \frac{(2m+1)!}{(m!)^2}\int_0^{\frac{x-1}{\delta}} t^m(1-t)^m dt.$$

We now compare the size of each Mellin transform. Rosser establishes (see [22, Theorem 15]) that

$$|I_{m,\delta}(\rho)| \le \frac{((1+\delta/m)^{m+1}+1)^m}{(\delta/m)^m|\gamma|^{m+1}} = \frac{2^m m^m}{\delta^m|\gamma|^{m+1}}(1+o(1)),$$

while we have from (2.7) and (3.5)

$$|F(\rho)| \le \frac{M(1,1+\delta,m)}{\delta^m|\gamma|^{m+1}} \le \frac{\sqrt{(2m)!(2m+1)!}}{m!\delta^m|\gamma|^{m+1}}(1+o(1)).$$

It follows from Stirling Formula that the quotient $\frac{|F(\rho)|}{|I_{m,\delta}(\rho)|} = \frac{\sqrt{(2m)!(2m+1)!}}{(2m)^m(m!)}$ decreases rapidly to 0 as m grows. For instance it is $0.0083\dots$ when we take $m=23$ for $b_0=50$.

3.4.2. *The new density of zeros.* When x is large enough, the largest contribution to $\Sigma(m,\delta,x)$ arises from

$$\sum_{\gamma>H,\sigma_0<\beta<1-\frac{1}{R\log\gamma}} \frac{x^{-\frac{1}{R\log\gamma}}}{\gamma^{m+1}}. \tag{3.11}$$

Rosser and successive authors took $\sigma_0 = 1/2$ since only bounds for $N(T)$ were available. Rosser and Schoenfeld find (see [24, equations (3.4), (3.16) and (2.4)]) that if $b_0 \le 2R\log^2 H$ and $x \ge e^{b_0}$ then

$$e^{\frac{b_0}{R\log H}} H^m \sum_{\gamma>H,1/2<\beta<1-\frac{1}{R\log\gamma}} \frac{x^{-\frac{1}{R\log\gamma}}}{\gamma^{m+1}} \le \frac{R(\log H)^3}{2\pi b_0\Big(\frac{mR(\log H)^2}{b_0}-1\Big)}(1+o(1)). \tag{3.12}$$

We are able to reduce significantly the contribution of the sum by using σ_0 closer to the limit of the zero-free region. We establish that if $b_0 \le 3R\log^2 H$ and $x \ge e^{b_0}$ then the above bound is replaced with

$$\Big(c_1 + c_2\frac{\log H}{H} + \frac{c_3}{H}\Big) + \Big(\big(c_1+\frac{c_2}{H}\big)\frac{R}{2b_0}\frac{(\log H)^2}{\big(\frac{mR}{b_0}\big)(\log H)^2-1}\Big).$$

When $\big(\frac{mR}{b_0}\big)(\log H)^2 - 1$ is large enough (for instance for $45 \le b_0 \le 2000$ and $m \ge 10$), the main contribution arises from the above left expression. We use the values for the c_i's from the right column of Table 2 as they make $c_1H + c_2\log H + c_3$ small. Otherwise, we use the values from the left column as they provide the smallest value for $c_1 + \frac{c_2}{H}$.

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TABLE 2. For all $T \geq H$, $N(\sigma, T) \leq c_1 T + c_2 \log T + c_3$.

	c_1 is small			$c_1 H + c_2 \log H + c_3$ is small		
σ	c_1	c_2	c_3	c_1	c_2	c_3
0.60	4.2288	2.2841	$-81\,673$	28.6424	2.2841	$-8.7674 \cdot 10^{11}$
0.65	2.4361	1.7965	$-97\,414$	17.1679	1.3674	$-5.2550 \cdot 10^{11}$
0.70	1.4934	1.4609	$-136\,370$	12.3778	0.9859	$-3.7888 \cdot 10^{11}$
0.75	1.0031	1.1442	$-169\,449$	9.6776	0.7708	$-2.9622 \cdot 10^{11}$
0.76	0.9355	1.0921	$-176\,604$	9.2730	0.7386	$-2.8384 \cdot 10^{11}$
0.77	0.8750	1.0437	$-184\,134$	8.9009	0.7089	$-2.7245 \cdot 10^{11}$
0.78	0.8205	0.9986	$-192\,120$	8.5575	0.6816	$-2.6194 \cdot 10^{11}$
0.79	0.7714	0.9566	$-200\,644$	8.2396	0.6563	$-2.5221 \cdot 10^{11}$
0.80	0.7269	0.9176	$-209\,795$	7.9445	0.6328	$-2.4317 \cdot 10^{11}$
0.81	0.6864	0.8812	$-219\,667$	7.6698	0.6109	$-2.3477 \cdot 10^{11}$
0.82	0.6495	0.8473	$-230\,367$	7.4135	0.5905	$-2.2692 \cdot 10^{11}$
0.83	0.6156	0.8157	$-242\,009$	7.1737	0.5714	$-2.1958 \cdot 10^{11}$
0.84	0.5846	0.7862	$-254\,724$	6.9490	0.5535	$-2.1270 \cdot 10^{11}$
0.85	0.5561	0.7586	$-268\,658$	6.7379	0.5367	$-2.0624 \cdot 10^{11}$
0.86	0.5297	0.7327	$-283\,978$	6.5392	0.5209	$-2.0016 \cdot 10^{11}$
0.87	0.5053	0.7085	$-300\,872$	6.3520	0.5059	$-1.9443 \cdot 10^{11}$
0.88	0.4827	0.6857	$-319\,555$	6.1751	0.4919	$-1.8901 \cdot 10^{11}$
0.89	0.4617	0.6644	$-340\,272$	6.0079	0.4785	$-1.8389 \cdot 10^{11}$
0.90	0.4421	0.6443	$-363\,301$	5.8494	0.4659	$-1.7905 \cdot 10^{11}$
0.91	0.4238	0.6253	$-388\,959$	5.6991	0.4539	$-1.7444 \cdot 10^{11}$
0.92	0.4066	0.6075	$-417\,606$	5.5564	0.4426	$-1.7007 \cdot 10^{11}$
0.93	0.3905	0.5906	$-449\,647$	5.4206	0.4318	$-1.6592 \cdot 10^{11}$
0.94	0.3754	0.5747	$-485\,543$	5.2913	0.4215	$-1.6196 \cdot 10^{11}$
0.95	0.3612	0.5596	$-525\,807$	5.1680	0.4116	$-1.5819 \cdot 10^{11}$
0.96	0.3478	0.5452	$-571\,018$	5.0503	0.4023	$-1.5458 \cdot 10^{11}$
0.97	0.3352	0.5316	$-621\,815$	4.9379	0.3933	$-1.5114 \cdot 10^{11}$
0.98	0.3232	0.5187	$-678\,911$	4.8304	0.3848	$-1.4785 \cdot 10^{11}$
0.99	0.3118	0.5063	$-743\,087$	4.7274	0.3766	$-1.4470 \cdot 10^{11}$

To verify the values for the c_i's, we refer the reader to [9, Section 6]: we choose the parameters from this article to be $H = H_0 - 1$, $\sigma_0 = 0.522817$ for $\sigma = 0.60$ and $\sigma_0 = 0.5208$ otherwise.

Table 3: Let $H = 3.061 \cdot 10^{10}$ and $b_0 \leq 9\,963$. For all $x \geq e^{b_0}$, $E(x) \leq \epsilon_0$.

b_0	σ_0	m	δ	T_1	ϵ_0
20	0.89	4	$1.363 \cdot 10^{-5}$	T_0	$5.3688 \cdot 10^{-4}$
25	0.89	3	$7.256 \cdot 10^{-6}$	T_0	$4.8208 \cdot 10^{-5}$
30	0.89	2	$2.811 \cdot 10^{-6}$	T_0	$5.6679 \cdot 10^{-6}$
35	0.91	3	$1.751 \cdot 10^{-7}$	16 739 408	$7.4457 \cdot 10^{-7}$
40	0.92	5	$2.142 \cdot 10^{-8}$	245 176 468	$8.6347 \cdot 10^{-8}$
45	0.92	13	$3.910 \cdot 10^{-9}$	4 085 373 679	$1.0358 \cdot 10^{-8}$
50	0.93	23	$3.116 \cdot 10^{-9}$	9 667 437 397	$2.3643 \cdot 10^{-9}$
55	0.93	24	$3.105 \cdot 10^{-9}$	10 162 544 235	$1.6783 \cdot 10^{-9}$
60	0.93	24	$3.099 \cdot 10^{-9}$	10 182 181 286	$1.6191 \cdot 10^{-9}$
65	0.94	24	$3.093 \cdot 10^{-9}$	10 201 894 453	$1.6114 \cdot 10^{-9}$
70	0.94	24	$3.087 \cdot 10^{-9}$	10 221 684 178	$1.6081 \cdot 10^{-9}$
75	0.94	24	$3.082 \cdot 10^{-9}$	10 238 234 420	$1.6052 \cdot 10^{-9}$
80	0.95	24	$3.225 \cdot 10^{-9}$	10 254 838 399	$1.6025 \cdot 10^{-9}$
85	0.95	24	$3.071 \cdot 10^{-9}$	10 274 834 474	$1.5997 \cdot 10^{-9}$
90	0.95	24	$3.066 \cdot 10^{-9}$	10 291 557 599	$1.5969 \cdot 10^{-9}$
95	0.95	24	$3.061 \cdot 10^{-9}$	10 308 335 305	$1.5942 \cdot 10^{-9}$
100	0.95	24	$3.056 \cdot 10^{-9}$	10 325 167 860	$1.5916 \cdot 10^{-9}$
200	0.97	23	$2.960 \cdot 10^{-9}$	10 175 863 512	$1.5422 \cdot 10^{-9}$
300	0.97	23	$2.866 \cdot 10^{-9}$	10 508 919 281	$1.4953 \cdot 10^{-9}$
400	0.98	22	$2.769 \cdot 10^{-9}$	10 360 124 846	$1.4476 \cdot 10^{-9}$
500	0.98	21	$2.674 \cdot 10^{-9}$	10 193 677 612	$1.4006 \cdot 10^{-9}$
600	0.98	20	$2.579 \cdot 10^{-9}$	10 015 840 574	$1.3543 \cdot 10^{-9}$
700	0.98	20	$2.492 \cdot 10^{-9}$	10 364 671 352	$1.3081 \cdot 10^{-9}$
800	0.98	19	$2.397 \cdot 10^{-9}$	10 181 118 220	$1.2616 \cdot 10^{-9}$
900	0.98	18	$2.303 \cdot 10^{-9}$	9 979 294 107	$1.2154 \cdot 10^{-9}$
1 000	0.98	17	$2.209 \cdot 10^{-9}$	9 761 696 912	$1.1695 \cdot 10^{-9}$
1 500	0.98	14	$1.753 \cdot 10^{-9}$	9 882 930 682	$9.3929 \cdot 10^{-10}$
2 000	0.99	10	$1.293 \cdot 10^{-9}$	9 091 299 627	$7.1125 \cdot 10^{-10}$
2 500	0.99	6	$8.300 \cdot 10^{-10}$	7 664 220 686	$4.8137 \cdot 10^{-10}$
3 000	0.99	2	$3.000 \cdot 10^{-10}$	4 992 468 020	$2.2211 \cdot 10^{-10}$
3 500	0.99	2	$9.200 \cdot 10^{-11}$	14 198 916 944	$6.6209 \cdot 10^{-11}$
4 000	0.99	2	$2.700 \cdot 10^{-11}$	26 575 655 437	$1.9689 \cdot 10^{-11}$
4 500	0.99	2	$7.810 \cdot 10^{-12}$	30 196 651 346	$5.8563 \cdot 10^{-12}$
5 000	0.99	2	$2.320 \cdot 10^{-12}$	30 572 809 972	$1.7434 \cdot 10^{-12}$
6 000	0.99	2	$2.100 \cdot 10^{-13}$	30 609 694 715	$1.5457 \cdot 10^{-13}$
7 000	0.99	2	$1.826 \cdot 10^{-14}$	30 609 997 695	$1.3693 \cdot 10^{-14}$
8 000	0.99	2	$1.618 \cdot 10^{-15}$	30 609 999 985	$1.2135 \cdot 10^{-15}$
9 000	0.99	2	$1.434 \cdot 10^{-16}$	H	$1.0755 \cdot 10^{-16}$
9 963	0.99	2	$1.390 \cdot 10^{-17}$	H	$9.5309 \cdot 10^{-18}$

For $45 \leq b_0 \leq 2000$ we use the values of c_i's from the right column of Table 2. We use the left values otherwise.

Table 4: Let $H = 2\,445\,999\,556\,030$ and $b_0 \le 13\,906$. For all $x \ge e^{b_0}$, $E(x) \le \epsilon_0$.

b_0	σ_0	m	δ	T_1	ϵ_0
20	0.88	4	$1.363 \cdot 10^{-5}$	T_0	$5.3688 \cdot 10^{-4}$
25	0.89	3	$7.256 \cdot 10^{-6}$	T_0	$4.8208 \cdot 10^{-5}$
30	0.89	2	$2.806 \cdot 10^{-6}$	T_0	$5.6646 \cdot 10^{-6}$
35	0.90	2	$1.604 \cdot 10^{-7}$	11 360 452	$7.0190 \cdot 10^{-7}$
40	0.91	3	$1.600 \cdot 10^{-8}$	174 242 715	$8.0214 \cdot 10^{-8}$
45	0.92	4	$1.613 \cdot 10^{-9}$	2 393 630 483	$8.6997 \cdot 10^{-9}$
50	0.93	7	$2.058 \cdot 10^{-10}$	36 960 925 828	$9.4602 \cdot 10^{-10}$
55	0.96	21	$5.079 \cdot 10^{-11}$	532 313 030 046	$1.1243 \cdot 10^{-10}$
60	0.96	28	$4.807 \cdot 10^{-11}$	770 935 427 426	$3.2156 \cdot 10^{-11}$
65	0.96	29	$4.801 \cdot 10^{-11}$	801 857 986 418	$2.5430 \cdot 10^{-11}$
70	0.96	29	$4.795 \cdot 10^{-11}$	802 859 999 396	$2.4849 \cdot 10^{-11}$
75	0.96	29	$4.789 \cdot 10^{-11}$	803 864 521 532	$2.4773 \cdot 10^{-11}$
80	0.97	29	$4.783 \cdot 10^{-11}$	804 871 562 262	$2.4738 \cdot 10^{-11}$
85	0.97	29	$4.777 \cdot 10^{-11}$	805 881 131 075	$2.4707 \cdot 10^{-11}$
90	0.97	29	$4.771 \cdot 10^{-11}$	806 893 237 503	$2.4677 \cdot 10^{-11}$
95	0.97	29	$4.765 \cdot 10^{-11}$	807 907 891 129	$2.4647 \cdot 10^{-11}$
100	0.97	29	$4.759 \cdot 10^{-11}$	808 925 101 582	$2.4618 \cdot 10^{-11}$
200	0.98	28	$4.647 \cdot 10^{-11}$	797 441 603 800	$2.4065 \cdot 10^{-11}$
300	0.98	28	$4.546 \cdot 10^{-11}$	815 133 603 120	$2.3543 \cdot 10^{-11}$
400	0.98	27	$4.440 \cdot 10^{-11}$	802 199 639 823	$2.3021 \cdot 10^{-11}$
500	0.98	26	$4.334 \cdot 10^{-11}$	788 664 950 273	$2.2506 \cdot 10^{-11}$
600	0.98	26	$4.237 \cdot 10^{-11}$	806 692 808 636	$2.1998 \cdot 10^{-11}$
700	0.99	25	$4.131 \cdot 10^{-11}$	792 643 976 191	$2.1480 \cdot 10^{-11}$
800	0.99	25	$4.032 \cdot 10^{-11}$	812 075 384 439	$2.0969 \cdot 10^{-11}$
900	0.99	23	$3.918 \cdot 10^{-11}$	762 588 970 852	$2.0443 \cdot 10^{-11}$
1 000	0.99	23	$3.818 \cdot 10^{-11}$	782 528 018 219	$1.9921 \cdot 10^{-11}$
1 500	0.99	20	$3.303 \cdot 10^{-11}$	774 756 126 279	$1.7342 \cdot 10^{-11}$
2 000	0.99	17	$2.788 \cdot 10^{-11}$	764 936 897 224	$1.4762 \cdot 10^{-11}$
2 500	0.99	14	$2.272 \cdot 10^{-11}$	752 424 086 843	$1.2118 \cdot 10^{-11}$
3 000	0.99	11	$1.755 \cdot 10^{-11}$	735 757 894 330	$9.5728 \cdot 10^{-12}$
3 500	0.99	7	$1.209 \cdot 10^{-11}$	618 567 513 247	$6.9073 \cdot 10^{-12}$
4 000	0.99	4	$6.800 \cdot 10^{-12}$	533 755 825 076	$4.2115 \cdot 10^{-12}$
4 500	0.99	2	$2.300 \cdot 10^{-12}$	576 348 240 050	$1.6858 \cdot 10^{-12}$
5 000	0.99	2	$8.400 \cdot 10^{-13}$	1 334 194 702 027	$6.0522 \cdot 10^{-13}$
6 000	0.99	2	$1.036 \cdot 10^{-13}$	2 401 904 005 983	$7.7686 \cdot 10^{-14}$
7 000	0.99	2	$1.332 \cdot 10^{-14}$	2 445 250 025 818	$9.9890 \cdot 10^{-15}$
8 000	0.99	2	$1.713 \cdot 10^{-15}$	2 445 987 153 821	$1.2845 \cdot 10^{-15}$
9 000	0.99	2	$2.202 \cdot 10^{-16}$	2 445 999 351 095	$1.6516 \cdot 10^{-16}$
10 000	0.99	2	$2.830 \cdot 10^{-17}$	2 445 999 552 648	$2.1236 \cdot 10^{-17}$
13 900	0.99	2	$9.502 \cdot 10^{-21}$	H	$7.1265 \cdot 10^{-21}$

We only use the values of c_i's from the left column of Table 2.

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