EXPLICIT ZERO-FREE REGIONS FOR DIRICHLET L-FUNCTIONS

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Abstract. Let $L(s,\chi)$ be the Dirichlet L-function associated to a non-principal primitive character χ modulo q with $3 \le q \le 400\,000$. We prove a new explicit zero-free region for $L(s,\chi)$: $L(s,\chi)$ does not vanish in the region $\mathfrak{Re} s \ge 1 - \frac{1}{R\log(q\max(1,|\mathfrak{Im} s|))}$ with $R = 5.60$. This improves a result of McCurley where 9.65 was shown to be an admissible value for R.

1. Introduction

In this article, we establish an explicit zero-free region for the Dirichlet L-functions associated to moduli for which the Generalized Riemann Hypothesis has been partially verified.

Theorem 1.1. *Let q be an integer with $3 \le q \le 400\,000$ and χ a non-principal primitive character modulo q. Then the Dirichlet L-function $L(s,\chi)$ does not vanish in the region:*

$$\mathfrak{Re} s \ge 1 - \frac{1}{5.60\log\left(q\max(1,|\mathfrak{Im} s|)\right)}. \tag{1.1}$$

This result improves previous results of [13] ($R = 6.436$) and [19] ($R = 9.646$). The case $q > 400\,000$ will be treated in a follow-up article. Theorem 1.1 uses ideas developed in [14] where it is proven that the Riemann zeta function does not vanish in the region

$$\mathfrak{Re} s \ge 1 - \frac{1}{5.70\log|\mathfrak{Im} s|},\ |\mathfrak{Im} s| \ge 2. \tag{1.2}$$

This improved long-standing results of Stechkin [27] and Rosser and Schoenfeld [26] (they proved that a value just under 9.65 was admissible). Since then, the method of [14] to prove (1.2) has been refined and 5.70 has been reduced to 5.68371 in [12] and to 5.573412 in [21].

Explicit zero-free regions for L-functions are useful to establish explicit results about the primes. For instance, (1.2) was used to obtain estimates for finite sums or products over the primes, locating primes in short intervals, applications to Diophantine problems, etc (see [4], [6], [7], [8], [17], [23], and [29]). Recently, Theorem 1.1 has already been applied to obtain new explicit bounds for $\psi(x;q,a)$ in [16] [3] (these works improve the well-known article [24] of Ramaré and Rumely). Some of the ideas to establish (1.2) and (1.1) have already been extended to Dedekind zeta functions [15] and Hecke L-functions [1]. We now present a quick overview of the main ingredients of the proof.

Let q be an integer with $3 \le q \le 400\,000$ and χ a non-principal primitive character modulo q. We consider $\varrho_0 = \beta_0 + i\gamma_0$ a non-trivial zero $(0 < \beta_0 < 1)$ of the Dirichlet L-function $L(s,\chi)$. Let s be a complex number with $\mathfrak{Re} s > 0$, and let denote κ and δ some positive parameters. Let f be a compactly supported, non-negative, "smooth" function (it has continuous derivatives up to a certain order). Instead of $-\mathfrak{Re}\frac{L'}{L}(s,\chi) = \mathfrak{Re}\sum_{n\ge1}\frac{\Lambda(n)\chi(n)}{n^s}$, we consider the "smoothed version"

$$\mathfrak{Re}\sum_{n\ge1}\frac{\Lambda(n)\chi(n)f(\log n)\left(1-\frac{\kappa}{n^\delta}\right)}{n^s}. \tag{1.3}$$

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We establish a version of Guinand-Weil's explicit formula of the form

$$\begin{aligned}\mathfrak{Re}\sum_{n\geq 1}\frac{\Lambda(n)\chi(n)f(\log n)\left(1-\frac{\kappa}{n^{\delta}}\right)}{n^s}&\\ =\frac{(1-\kappa)}{2}f(0)\log(q|\mathfrak{Im}s|)&-\sum_{\varrho\in Z(\chi)}\mathfrak{Re}\left(F(s-\varrho)-\kappa F(s+\delta-\varrho)\right)+E_q(s),\end{aligned}\tag{1.4}$$

where F is the Laplace transform of f, χ is non-principal, $Z(\chi)$ is the set of non-trivial zeros of $L(s,\chi)$, and $E_q(s)$ is an error term. In addition, when χ is principal, the term $a_0\mathfrak{Re}F(s-1)$ arises for $k=0$ from the pole of $\zeta(s)$ at $s=1$. To compare with the classical proof, κ and δ would each be 0, f would be identically 1, $\mathfrak{Re}F(s-1)$ would be $\frac{1}{\mathfrak{Re}s-1}$, and $-\sum_{\varrho\in Z(\chi)}\mathfrak{Re}\frac{1}{s-\varrho}$ would be the sum over the zeros. We apply (1.3) at various values of s on a vertical line passing near ϱ_0. In particular we take $s=\sigma+ik\gamma_0$ with the integer k ranging between 0 and a fixed integer n_0. By means of a trigonometric inequality of the form

$$P(t)=\sum_{k=0}^{n_0}a_k\cos(kt)\geq 0 \quad\text{with}\quad a_k\geq 0 \quad\text{for all}\quad k=0,\dots,n_0,$$

we deduce

$$\sum_{n\geq 1}\frac{\Lambda(n)f(\log n)\left(1-\frac{\kappa}{n^{\delta}}\right)}{n^{\sigma}}\sum_{k=0}^{n_0}a_k\cos\left(k\arg\left(\frac{\chi(n)}{n^{i\gamma_0}}\right)\right)\geq 0.$$

It remains to give accurate upper bounds to the right hand side of (1.4) for each $s=\sigma+ik\gamma_0$. First we isolate the term arising from $k=1$ and the zero ϱ_0 in the sum over the zeros. In the classical proof, the sum over the remaining zeros is simply discarded as $\mathfrak{Re}\frac{1}{s-\varrho}$ is always positive assuming $\mathfrak{Re}s>1$. The situation here is more complicated as we are considering the difference $\mathfrak{Re}\left(F(s-\varrho)-\kappa F(s+\delta-\varrho)\right)$ and as we are allowing s to be located inside the critical strip. The main idea of Stechkin [27] was to impose conditions of κ,δ to deal with $\mathfrak{Re}\left(\frac{1}{s-\varrho}-\frac{\kappa}{s+\delta-\varrho}\right)$. This trick was then exploited by Rosser and Schoenfeld in [26] and by McCurley in [19]. We are using here the version adapted to smooth functions from [14]: after imposing on f,κ and δ some appropriate conditions, we show $\sum_{\substack{\varrho\in Z(\chi)\\ \mathfrak{Re}\varrho\leq\mathfrak{Re}s}}\mathfrak{Re}\left(F(s-\varrho)-\kappa F(s+\delta-\varrho)\right)\geq 0$. We then prove that the size of the sum over the remaining zeros is negligible. This argument allows us to multiply the final constant in the zero-free region by a factor of $\frac{1-\kappa}{2}\simeq\frac{1-\frac{1}{\sqrt{5}}}{2}\simeq 0.28$. Note that this argument is valid for all moduli q. On the other hand, the Burgess bound argument used by Heath-Brown in [9] leads to a coefficient of 0.25 but was only valid for sufficiently large moduli. Putting together all these arguments leads to the inequality:

$$\frac{1-\kappa}{2}f(0)\log\left(q|\gamma_0|\right)\sum_{k=1}^{n_0}a_k-a_1F(\sigma-\beta_0)+a_0F(\sigma-1)+\epsilon\geq 0,\tag{1.5}$$

where ϵ is an error term. We choose f to depend on β_0 by setting $f(0)=h(0)(1-\beta_0)$, where $h(0)$ is independent of ϱ_0 and h is a smooth function chosen appropriately. We also choose the polynomial coefficients a_i, and the parameter σ. Then the inequality

$$(1-\beta_0)\log\left(q|\gamma_0|\right)\geq\frac{a_1F(\sigma-\beta_0)-a_0F(\sigma-1)-\epsilon}{\frac{1-\kappa}{2}h(0)\sum_{k=1}^{n_0}a_k}\tag{1.6}$$

provides a formula where the zero-free constant R^{-1} is given by the term on the right.

2. Explicit results about the zeros of Dirichlet L-functions

We list here the most recent results for zeros of Dirichlet L-functions which shall be applied in this article.

2.1. **Partial numerical verification of GRH.** In 2013, Platt provided a partial numerical verification of the Generalized Riemann Hypothesis which asserts that all zeros of Dirichlet L-functions inside the critical strip $0 < \mathfrak{Re} s < 1$ lie on the vertical $1/2$-line:

Theorem 2.1. [22, Theorem 7.1] *GRH holds for Dirichlet L-functions of primitive characters modulo $3 \le q \le 400\,000$ and to height $H_q = \max\left(\frac{10^8}{q}, \frac{c_q \cdot 10^7}{q} + 200\right)$ with $c_q = 7.5$ if q is even and 3.75 otherwise.*

This improves drastically both numerically and theoretically on Rumely's work (see [24] and [2]). It increases by a factor of $1\,000$ the number of moduli for which the verification was undertaken, and it increases by a factor of between 100 and $10\,000$ the size of the height H_q (for comparable moduli). This theorem has many important applications and has already contributed to Helfgott's proof of the Ternary Goldbach Conjecture [10] and [11].

2.2. **Explicit zero-free regions.**

Theorem 2.2. [19, Theorem 1] *Let $\mathfrak{L}_q(s)$ be the product of the $\phi(q)$ Dirichlet L-functions formed with characters modulo q. Let $M = \max\{q, q|\mathfrak{Im} s|, 10\}$ and $R = 9.645908801$. Then $\mathfrak{L}_q(s)$ has at most a single zero in the region*

$$\left\{ s : \mathfrak{Re} s \ge 1 - \frac{1}{R \log M} \right\}.$$

The only possible zero in this region is a simple real zero arising from an L-function formed with a real non-principal character modulo q.

In the PhD dissertation [13] it was established that 6.4355 is an admissible constant for R.

2.3. **Explicit estimate of the number of zeros in a box.** We denote $N(T,\chi)$ be the number of zeros of $L(s,\chi)$ in the rectangle $0 < \mathfrak{Re} s < 1$ and $|\mathfrak{Im} s| < T$.

Theorem 2.3. *Let χ be a primitive non-principal character modulo q. Then there exist $C_1, C_2 > 0$ s.t. for every $T \ge 10$*

$$\left| N(T,\chi) - \frac{T}{\pi} \log\left(\frac{qT}{2\pi e}\right) \right| \le C_1 \log(qT) + C_2.$$

In other words:

$$N_2(T,q) \le N(T,\chi) \le N_1(T,q), \quad \text{with} \quad N_j(T,q) = \frac{T}{\pi} \log \frac{qT}{2\pi e} + (-1)^{j+1} \left(C_1 \log(qT) + C_2 \right).$$

Theorem 2.3 was first established by McCurley [19, Theorem 2.1]. We use here Trudgian's version [28, Theorem 1] with

$$C_1 = 0.247, \; C_2 = 8.949. \tag{2.1}$$

3. Setting up the argument

3.1. **Notation.** Let q be a modulus for which the verification of the GRH up to height H_q has been established. In this article, we use Theorem 2.1 and assume that

$$2 \le q \le 400\,000, \; H_q > 293, \quad \text{and} \quad qH_q \ge Q_0 = 10^8.$$

Let q_0, q_1 be positive fixed integers so that $q_0 \le q \le q_1$. For computational purposes we split in two cases $3 \le q \le 1\,000$ and $1\,001 \le q \le 400\,000$, so $q_0 = 3, q_1 = 1\,000, H_q \ge 100\,000$ and $q_0 = 1\,000, q_1 = 400\,000, H_q \ge 293$ respectively.

Let χ be a non-principal primitive character of conductor q and let $\varrho_0 = \beta_0 + i\gamma_0$ be a non-trivial zero of $L(s,\chi)$ satisfying

$$\gamma_0 \geq H_q, \ q\gamma_0 \geq Q_0, \quad \text{and} \quad \beta_0 < 1 - \frac{1}{R\log\left(q|\mathfrak{Im}\gamma_0|\right)} \quad \text{with} \quad R = 9.646. \tag{3.1}$$

We introduce the parameters r and η such that

$$5 \leq r \leq R \text{ and } \eta = 1 - \beta_0 = \frac{1}{r\log(q\gamma_0)}. \tag{3.2}$$

Note that $r < 5$ proves a zero-free region with admissible constant 5.
Let P be a trigonometric polynomial of degree $n_0 \geq 2$ satisfying

$$P(x) = \sum_{k=0}^{n_0} a_k \cos(kx) \geq 0, \quad \text{with} \quad a_k \geq 0, \text{ and } 0 < a_0 < a_1. \tag{3.3}$$

We denote $A = \sum_{k=1}^{n_0} a_k$. We refer to Section 5.6 for the explicit definition of the polynomial used in this article. Here we use a polynomial of degree $n_0 = 16$.
Let $t_0 > 0$ (here $t_0 = 100$) and let

$$\sigma = 1 - \frac{1}{R\log(q(n_0\gamma_0 + t_0))}, \tag{3.4}$$

$$\omega = \omega(\eta, q) = \frac{1-\sigma}{\eta} = \frac{1}{R\eta\log\left(q(n_0\gamma_0 + t_0)\right)} = \frac{1}{R\eta\log\left(n_0 e^{\frac{1}{r\eta}} + qt_0\right)}. \tag{3.5}$$

Since $q\gamma_0 \geq Q_0$, we have the ranges

$$\max(\beta_0, \sigma_0) < \sigma < 1 \quad \text{with } \sigma_0 = 1 - \frac{1}{R\log\left(Q_0 n_0 + q_0 t_0\right)}, \tag{3.6}$$

and

$$0 < \eta \leq \eta_0 \quad \text{with } \eta_0 = \frac{1}{r\log Q_0}. \tag{3.7}$$

Let $\eta_1 \in (0, \eta_0)$. We have

$$\begin{cases} \omega_1 \leq \omega \leq \frac{r}{R} & \text{when} \quad 0 < \eta \leq \eta_1, \\ \omega_0 \leq \omega \leq \omega_1 & \text{when} \quad \eta_1 < \eta \leq \eta_0, \end{cases} \tag{3.8}$$

$$\text{with} \quad \omega_0 = \omega(\eta_0, q_1) = \frac{r\log Q_0}{R\log\left(n_0 Q_0 + t_0 q_1\right)} \quad \text{and} \quad \omega_1 = \omega(\eta_1, q_0) = \frac{1}{R\eta_1\log\left(n_0 e^{\frac{1}{r\eta_1}} + q_0 t_0\right)}. \tag{3.9}$$

Let δ, κ be some parameters satisfying

$$0.5 < \delta < 0.75, \text{ and } 0.25 < \kappa < \min\left(\frac{\sigma_0}{1+\delta}, \frac{1}{2\delta+1}\right). \tag{3.10}$$

In addition, we assume

$$\max\left(10, \left((\kappa(2\delta+1))^{-1} - 1\right)^{-1/2}\right) < t_0 < 293 < H_q. \tag{3.11}$$

We provide in Proposition 5.2 the definitions of κ and δ depending on h, σ_0, and η_0.
Finally we add the condition

$$a_0 + \left(\frac{1-\kappa}{\max\left(\frac{1}{2^{\sigma_0}-1} - \frac{\kappa}{2^{\sigma_0+\delta}-1}, 2\left(\frac{1}{3^{\sigma_0}-1} - \frac{\kappa}{3^{\sigma_0+\delta}-1}\right)\right)} - 1\right)\sum_{k=2}^{n_0} a_k > 0. \tag{3.12}$$

Numerical data for all the above parameters (r, κ, δ) are given in Tables 2 and 3, Section 5.7.

3.2. **Introducing a smooth weight.** We now introduce the weight f which is used in the study of $\sum_{n\geq 1}\frac{\chi(n)\Lambda(n)}{n^s}f(\log n)\left(1-\frac{\kappa}{n^\delta}\right)$. We define

$$f(t)=f_\eta(t)=\eta h(\eta t), \tag{3.13}$$

where h is a function satisfying:

$$\begin{cases} h \text{ is compactly supported in } [0,d], \text{ for some } \ d>0, \\ h \text{ is positive in } [0,d), \\ h\in\mathcal{C}^2([0,d]), \\ h(d)=h'(0)=h'(d)=h''(d)=0. \end{cases} \tag{3.14}$$

Note that in [14], we used the notation $g_1(\theta)$ and m instead of $h(0)$ and m_h respectively. We give more details in Section 5.5 on the motivation for the explicit choice for f as well as study its analytical properties.

We denote F the Laplace transform of f:

$$F(s)=\int_0^\infty e^{-st}f(t)dt=\int_0^d e^{-\frac{su}{\eta}}h(u)du, \tag{3.15}$$

F_2 the Laplace transform of f'', and

$$H(x,y)=\mathfrak{Re}\left(\frac{F_2(x+iy)}{(x+iy)^2}\right). \tag{3.16}$$

We impose a last and important condition on f: that its Laplace transform satisfies

$$\mathfrak{Re}F(z)\geq 0 \quad\text{when}\quad \mathfrak{Re}z\geq 0. \tag{3.17}$$

Remark. This is achieved by choosing h as a self convolution: $h=g\star g$, and we refer the reader to [9, Section 7] for details. In the classical proof, property (3.17) is satisfied by $\frac{1}{s}$ and is key in handling the sum over the zeros.

We now recall some properties established in [14, Lemma 3.2]

Lemma 3.1. *Let x,y be real numbers satisfying $x+iy\neq 0$.*

$$\mathfrak{Re}F(x+iy)=\frac{x}{x^2+y^2}h(0)\eta+H(x,y), \tag{3.18}$$

$$\text{where}\quad |H(x,y)|\leq\frac{M\left(x/\eta\right)}{x^2+y^2}\eta^2,\ \text{ with } M(z)=\int_0^d\left|h''(u)\right|e^{-zu}du. \tag{3.19}$$

In addition, when $x\geq 0$, then

$$|H(x,y)|\leq\frac{m_h}{x(x^2+y^2)}\eta^3,\ \text{ with } m_h=\max_{u\in[0,d]}|h''(u)|. \tag{3.20}$$

Remark. We choose h among several choices of families of functions. The explicit definition of h eventually depends on several extra parameters (see Section 5.5).

3.3. **An explicit formula for a smoothed version of $-\frac{L'}{L}(s,\chi)$.**

Proposition 3.2. *Let f be a function satisfying* (3.14)*. Let s be a complex number, and χ a primitive (non-principal) character modulo q. Then*

$$\mathfrak{Re}\sum_{n\geq 1}\frac{\Lambda(n)\chi(n)f(\log n)}{n^s}=-\mathfrak{Re}\sum_{\varrho\in Z(\chi)}F(s-\varrho)+\frac{f(0)}{2}\left(\log\frac{q}{\pi}+\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)\right)$$
$$+\mathfrak{Re}\frac{1}{2i\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{z+\mathfrak{a}}{2}\right)\frac{F_2(s-z)}{(s-z)^2}dz, \tag{3.21}$$

$$\mathfrak{Re}\sum_{n\geq 1}\frac{\Lambda(n)f(\log n)}{n^s} = \mathfrak{Re}F(s-1) - \sum_{\varrho\in Z(\zeta)}\mathfrak{Re}F(s-\varrho) + \frac{f(0)}{2}\left(-\log\pi + \mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)\right)$$
$$+\mathfrak{Re}\frac{F_2(s)}{s^2} + \mathfrak{Re}\frac{1}{2i\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{z}{2}\right)\frac{F_2(s-z)}{(s-z)^2}dz, \quad (3.22)$$

where $Z(\zeta)$ and $Z(\chi)$ are the sets of non-trivial zeros of $\zeta(s)$ and of $L(s,\chi)$, respectively.

Proof of Proposition 3.2 is postponed until Section 6.2. The case of the principal character χ_0 follows from (3.22) and the identity

$$\mathfrak{Re}\sum_{n\geq 1}\frac{\Lambda(n)\chi_0(n)f(\log n)}{n^s} = \mathfrak{Re}\sum_{n\geq 1}\frac{\Lambda(n)f(\log n)}{n^s} - \mathfrak{Re}\sum_{m\geq 1}\sum_{p|q}\frac{(\log p)f(m\log p)}{p^{ms}}.$$

Let $s = x+iy$ be a complex number. For κ,δ as defined in (3.10), we introduce

$$S(s) = \mathfrak{Re}\sum_{n\geq 1}\frac{\Lambda(n)f(\log n)}{n^s}\left(1-\frac{\kappa}{n^\delta}\right), \quad (3.23)$$

$$S(s,\chi) = \mathfrak{Re}\sum_{n\geq 1}\frac{\chi(n)\Lambda(n)f(\log n)}{n^s}\left(1-\frac{\kappa}{n^\delta}\right). \quad (3.24)$$

$$E(x,y) = H(x,y) - \kappa H(x+\delta,y). \quad (3.25)$$

$$D(s) = \mathfrak{Re}(F(s)-\kappa F(s+\delta)) = \left(\frac{x}{x^2+y^2} - \frac{\kappa(x+\delta)}{(x+\delta)^2+y^2}\right)h(0)\eta + E(x,y), \quad (3.26)$$

$$\mathfrak{G}_{\kappa,\delta}(s) = \mathfrak{Re}\frac{\Gamma'}{\Gamma}(s/2) - \kappa\mathfrak{Re}\frac{\Gamma'}{\Gamma}((s+\delta)/2). \quad (3.27)$$

In addition we define

$$D_1(s) = \frac{h(0)\eta}{2}\left[-(1-\kappa)\log\pi + \mathfrak{G}_{\kappa,\delta}(s+2)\right], \quad (3.28)$$

$$D_1(s,\chi) = \frac{h(0)\eta}{2}\left[(1-\kappa)\log\frac{q}{\pi} + \mathfrak{G}_{\kappa,\delta}(s+\mathfrak{a})\right], \quad (3.29)$$

$$D_2(s) = E(x,y) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathfrak{Re}\frac{\Gamma'}{\Gamma}((1/2+it)/2)E(x-1/2,y-t)dt, \quad (3.30)$$

$$D_2(s,\chi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathfrak{Re}\frac{\Gamma'}{\Gamma}((1/2+\mathfrak{a}+it)/2)E(x-1/2,y-t)dt. \quad (3.31)$$

Using this notation and noting that $f(0) = h(0)\eta$ and $\mathfrak{Re}\frac{\Gamma'}{\Gamma}(s) = \mathfrak{Re}\frac{\Gamma'}{\Gamma}(s/2)$, the explicit formulae from Proposition 3.2 become

$$S(s) = D(s-1) - \sum_{\varrho\in Z(\zeta)}D(s-\varrho) + D_1(s) + D_2(s), \quad (3.32)$$

$$S(s,\chi) = -\sum_{\varrho\in Z(\chi)}D(s-\varrho) + D_1(s,\chi) + D_2(s,\chi). \quad (3.33)$$

3.4. **An explicit inequality.** Taking $x = \arg\left(\frac{\chi(n)}{n^{i\gamma_0}}\right)$ in the non-negative trigonometric inequality (3.3), we get

$$\sum_{k=0}^{n_0}a_k S(\sigma+ik\gamma_0,\chi^k)\geq 0. \quad (3.34)$$

Note that Proposition 3.2 can be applied to χ^k only when it is a non-primitive character. So for each $k \geq 2$, we introduce q_k the conductor for the Dirichlet character χ^k, and $\chi_{(k)}$ the unique primitive character modulo q_k that induces χ^k. (We refer the reader to [20, Theorem 9.2.] and recall that for $k=0$, $\chi_{(0)}$ is identically 1, and for $k=1, \chi_{(1)}=\chi$.) We establish in Lemma 6.2 that under condition (3.12), then

$$\sum_{k=0}^{n_0} a_k S(\sigma+ik\gamma_0,\chi_{(k)}-\chi^k)+\frac{1-\kappa}{2}f(0)\sum_{k=1}^{n_0} a_k \log\left(\frac{q}{q_k}\right)\geq 0. \tag{3.35}$$

Together with (3.34) and (3.35), we obtain

$$\sum_{k=0}^{n_0} a_k S(\sigma+ik\gamma_0,\chi_{(k)})+\frac{1-\kappa}{2}f(0)\sum_{k=2}^{n_0} a_k \log\left(\frac{q}{q_k}\right)\geq 0, \tag{3.36}$$

and we can now apply the explicit formulae from Proposition 3.2 to each "primitive term" $S(\sigma+ik\gamma_0,\chi_{(k)})$. We introduce the notation

$$\mathcal{E}_1 = a_0 D(\sigma-1) = a_0\left(F(\sigma-1,0)-\kappa F(\sigma-1+\delta,0)\right), \tag{3.37}$$

$$\mathcal{E}_2 = \sum_{k=0}^{4} a_k \sum_{\varrho\in Z(\chi_{(k)})} D(\sigma+ik\gamma_0-\varrho), \tag{3.38}$$

$$\begin{aligned}\mathcal{E}_3 &= a_0 D_1(\sigma)+\sum_{k=1}^{n_0} a_k\left(D_1(\sigma+ik\gamma_0,\chi_{(k)})+\frac{1-\kappa}{2}f(0)\log(q/q_k)\right) \qquad (3.39)\\ &= \frac{h(0)\eta}{2}\Big[a_0\left(-(1-\kappa)\log\pi+\mathfrak{G}_{\kappa,\delta}(\sigma+2)\right)+(1-\kappa)A\log(q/\pi)+\sum_{k=1}^{n_0} a_k\mathfrak{G}_{\kappa,\delta}(\sigma+\mathfrak{a}_k+ik\gamma_0)\Big],\end{aligned}$$

$$\begin{aligned}\mathcal{E}_4 &= a_0 D_2(\sigma)+\sum_{k=1}^{n_0} a_k D_2(\sigma+ik\gamma_0,\chi_{(k)}) \qquad (3.40)\\ &= a_0 E(\sigma,0)+\sum_{k=0}^{n_0}\frac{a_k}{2\pi}\int_{-\infty}^{+\infty}\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1/2+\mathfrak{a}_k+it}{2}\right)E(\sigma-1/2,k\gamma_0-t)dt,\end{aligned}$$

where $\mathfrak{a}_k$ is associated to the character χ_k:

$$\mathfrak{a}_k=\frac{1-\chi_k(-1)}{2}. \tag{3.41}$$

In particular $\mathfrak{a}_0=0$. Using this notation to rewrite explicit formulae (3.32) and (3.33) respectively, we rewrite (3.36) more simply as

$$0\leq \mathcal{E}_1-\mathcal{E}_2+\mathcal{E}_3+\mathcal{E}_4. \tag{3.42}$$

4. Sketch of the proof of the Zero-Free Region

4.1. Studying each $\mathcal{E}_i$. We make use of Lemma 3.1 for this and refer the reader to the next section for full details.

- For $\mathcal{E}_1$, the pole of the Riemann zeta function at $s=1$ gives the contribution $a_0D(\sigma-1)=a_0F(\sigma-1)+\mathcal{O}(\eta)=\frac{a_0}{\sigma-1}+\mathcal{O}(\eta)$. We prove in Lemma 5.1 that
$$\mathcal{E}_1\leq a_0F(\sigma-1)+\mathcal{C}_1(\eta).$$
- In the sum over the zeros $\mathcal{E}_2$, we isolate the term at $k=1, \varrho=\beta_0+i\gamma_0$:
$$a_1\left(F(\sigma-\beta_0)-\kappa F(\sigma+\delta-\beta_0)\right)=a_1F(\sigma-\beta_0)+\mathcal{O}(\eta).$$

The positivity property (3.17) for the Laplace transform F allows us to discard most of the zeros on the right of the vertical line at σ. This is done by means of a generalization of Stechkin's lemma (see Section 5.2.2 and in particular Proposition 5.2):
for d the solution of the equation

$$(2t+1)h(0)+\left(\frac{1}{t}+\frac{1}{2\sigma_0-1+t}\right)m_h\eta_0^2 \\ =\left(\frac{1}{t}+\frac{1+t}{(2\sigma_0-1+t)^2}\right)h(0)+\left(\frac{1}{t^3}+\frac{1}{(2\sigma_0-1+t)^3}\right)m_h\eta_0^2 \quad (4.1)$$

and for

$$\kappa=\frac{(2\sigma_0-1)h(0)-\frac{m_h\eta_0^2}{2\sigma_0-1}}{(2\delta+1)h(0)+\left(\frac{1}{\delta}+\frac{1}{2\sigma_0-1+\delta}\right)m_h\eta_0^2}, \quad (4.2)$$

then

$$D(\sigma-\beta+iy)+D(\sigma-1+\beta+iy)\geq 0$$

as soon as $1/2<\beta<\sigma$ and $y>0$.
Finally, we use an explicit zero density estimate to prove that the remaining sum contributes negligibly. We prove in Lemma 5.4 that

$$\mathcal{E}_2\geq a_1F(\sigma-\beta_0,0)+\mathcal{C}_2(\eta). \quad (4.3)$$

- We use Stirling's formula to estimate the Gamma terms appearing in $\mathcal{E}_3$ and $\mathcal{E}_4$. Since $\gamma_0>1$, we have $|\mathfrak{G}_{\kappa,\delta}(\sigma+\mathfrak{a}_k+ik\gamma_0)|\leq\log\gamma_0+\mathcal{O}(1)$, for all $k\geq 1$. We prove in Lemma 5.5 that
$$\mathcal{E}_3\leq\frac{A(1-\kappa)h(0)}{2}\log(q\gamma_0)\eta+\mathcal{C}_3(\eta).$$
- Finally, the integral term is negligible and we prove in Lemma 5.8 that
$$\mathcal{E}_4\leq\mathcal{C}_4(\eta).$$

The positivity argument (3.42) can then be rewritten as

$$0\leq a_0F(\sigma-1)-a_1F(\sigma-\beta_0)+\frac{A(1-\kappa)h(0)}{2}\log(q\gamma_0)\eta+\mathfrak{e}(\eta), \quad (4.4)$$

where the error term is

$$\mathfrak{e}(\eta)=\mathcal{C}_1(\eta)+\mathcal{C}_2(\eta)+\mathcal{C}_3(\eta)+\mathcal{C}_4(\eta). \quad (4.5)$$

We introduce the difference

$$K_h(\omega)=a_1F(\sigma-\beta_0)-a_0F(\sigma-1)=\int_0^d\left(a_1e^{-t}-a_0\right)h(t)e^{\omega t}dt, \quad (4.6)$$

so that inequality (4.4) becomes

$$r=\frac{1}{\log(q\gamma_0)\eta}\leq\frac{A(1-\kappa)h(0)}{2(K_h(\omega)-\mathfrak{e}(\eta))}. \quad (4.7)$$

4.2. **A strategy to compute the constant in the zero-free-region.** It remains to find a lower bound as large as possible for $K_h(\omega)-\mathfrak{e}(\eta)$ for the values of η in $(0,\eta_0)$.
Since $\frac{\partial K_h}{\partial\omega}(\omega)=\int_0^d\left(a_1e^{-t}-a_0\right)h(t)te^{\omega t}dt$, with $a_1e^{-t}-a_0>0$ when $t<\log\frac{a_1}{a_0}$ and $\omega_0\leq\omega\leq\frac{r}{R}$,

then $\frac{\partial K_h}{\partial \omega}(\omega) > \int_0^{\log(a_1/a_0)} \left(a_1 e^{-t} - a_0\right) h(t) t e^{\omega_0 t} dt + \int_{\log(a_1/a_0)}^{d} \left(a_1 e^{-t} - a_0\right) h(t) t e^{\frac{r}{R} t} dt$. We choose values for ω_0 so that

$$\int_0^{\log(a_1/a_0)} \left(a_1 e^{-t} - a_0\right) h(t) t e^{\omega_0 t} dt + \int_{\log(a_1/a_0)}^{d} \left(a_1 e^{-t} - a_0\right) h(t) t e^{\frac{r}{R} t} dt > 0 \tag{4.8}$$

and $K_h(\omega)$ increases with $\omega \in (\omega_0, r/R)$.
From the definitions of the $\mathcal{C}_i$'s in (5.2), (5.23), (5.25), and (5.40), we have

$$\mathfrak{e}(\eta) = \eta(\alpha_1 + \alpha_2 \eta + \alpha_3 \eta^2), \tag{4.9}$$

where the α_i's are computable constants and satisfy $\alpha_1 < 0, \alpha_2 > 0, \alpha_3 > 0$. We have that $\mathfrak{e}(\eta)$ is negative and decreases from $\eta = 0$ to $\eta_2 = \frac{-\alpha_2 + \sqrt{\alpha_2^2 - 3\alpha_1\alpha_3}}{3\alpha_3}$, then it increases (becoming positive after the root $\frac{-\alpha_2 + \sqrt{\alpha_2^2 - 4\alpha_1\alpha_3}}{2\alpha_3}$).
We are now able to establish a lower bound for $K_h(\omega) - \mathfrak{e}(\eta)$. We fix $\eta_1 \in (0, \eta_0)$ and bound $K_h(\omega) - \mathfrak{e}(\eta)$ in each following cases, depending on the location of η_1 with respect to η_2. We denote

$$\omega_1 = \omega(\eta_1, q_0),\ \omega_2 = \omega(\eta_2, q_0). \tag{4.10}$$

(1) If $0 < \eta_0 < \eta_2$, then

$$K_h(\omega) - \mathfrak{e}(\eta) > \begin{cases} K_h(\omega_1) & \text{if } 0 < \eta < \eta_1, \\ K_h(\omega_0) - \mathfrak{e}(\eta_1) & \text{if } \eta_1 < \eta < \eta_0. \end{cases}$$

(2) If $0 < \eta_2 < \eta_0$ and $0 < \eta_1 < \eta_2$ then

$$K_h(\omega) - \mathfrak{e}(\eta) > \begin{cases} K_h(\omega_1) & \text{if } 0 < \eta < \eta_1, \\ K_h(\omega_0) - \max(\mathfrak{e}(\eta_1), \mathfrak{e}(\eta_0)) & \text{if } \eta_1 < \eta < \eta_0. \end{cases}$$

(3) If $0 < \eta_2 < \eta_0$ and $\eta_2 < \eta_1 < \eta_0$ then

$$K_h(\omega) - \mathfrak{e}(\eta) > \begin{cases} K_h(\omega_1) & \text{if } 0 < \eta < \eta_1, \\ K_h(\omega_0) - \mathfrak{e}(\eta_0) & \text{if } \eta_1 < \eta < \eta_0. \end{cases}$$

We deduce that, for all $\eta \in (0, \eta_0)$, $K_h(\omega) - \mathfrak{e}(\eta) > \mathcal{K}_1$, with

$$\mathcal{K}_1 = \begin{cases} \min\left(K_h(\omega_1), K_h(\omega_0) - \mathfrak{e}(\eta_1)\right) & \text{if } 0 < \eta_0 < \eta_2, \\ \min\left(K_h(\omega_1), K_h(\omega_0) - \max(\mathfrak{e}(\eta_1), \mathfrak{e}(\eta_0))\right) & \text{if } 0 < \eta_2 < \eta_0. \end{cases} \tag{4.11}$$

Finally, we choose the value for $\eta_1 < \eta_0$ so as to make $\mathcal{K}_1$ as large as possible within the constraints of (3.2), (3.3), (3.10), (3.11), (3.12), and (3.14). With this value for $\mathcal{K}_1$, we define the constant in the zero-free region by

$$\frac{A(1-\kappa)h(0)}{2\mathcal{K}_1}. \tag{4.12}$$

Details for the computations can be found in Section 5.7.
Remark. In [14], values for η_0 were chosen so that $\mathfrak{e}(\eta_0) < 0$. This allowed us to take

$$\frac{A(1-\kappa)h(0)}{2K_h(\omega_0)}$$

as an admissible value for the constant in the zero-free region for the Riemann zeta function. The main improvement in [21] comes from refining this argument by dividing the interval of study at a value η_1 between 0 and η_0. The constant is then given by

$$\frac{A(1-\kappa)h(0)}{2} \min_{0<\eta_1\leq\eta_0} \max\left(\frac{1}{K_h(\omega_1)}, \frac{1}{K_h(\omega_0) - \mathfrak{e}(\eta_1)}\right),$$

assuming $\mathfrak{e}(\eta)$ is decreasing on $(0,\eta_0)$ and $\eta_0 \le \eta_2$.

5. Details of the proof of the zero-free region

5.1. Study of polar term $\mathcal{E}_1$.

Lemma 5.1. *Let $\kappa,\delta>0$, and $0<\sigma<1$. Then*

$$\mathcal{E}_1 \le a_0 F(\sigma-1,0) + \mathcal{C}_1(\eta), \tag{5.1}$$

$$\text{where } \ \mathcal{C}_1(\eta) = a_0\left(-\frac{h(0)\kappa}{\delta}\eta + \frac{m_h\kappa}{(\sigma_0-1+\delta)^3}\eta^3\right). \tag{5.2}$$

Proof. We use (3.18) and (3.20) to bound $F(\sigma-1+\delta)$:

$$\left|F(\sigma-1+\delta,0) - \frac{h(0)}{\sigma-1+\delta}\eta\right| \le |H(\sigma-1+\delta,0)| \le \frac{m_h}{(\sigma-1+\delta)^3}\eta^3.$$

□

5.2. Study of the sum over the zeros $\mathcal{E}_2$.

Let χ be a primitive character modulo q and let $k=0,\dots,n$. We recall that $\chi_{(k)}$ is the primitive character induced by χ^k, q_k is its conductor, $Z(\chi_{(k)})$ is the set of non-trivial zeros of $L(s,\chi_{(k)})$, and in particular $Z(\chi_{(0)})$ is the set of non-trivial zeros of the Riemann zeta function $\zeta(s)$. We study here

$$\mathcal{E}_2 = \sum_{k=0}^{n_0} a_k \sum_{\varrho\in Z(\chi_{(k)})} D(\sigma+ik\gamma_0-\varrho). \tag{5.3}$$

Using the symmetry of the zeros with respect to the critical line, we write

$$\sum_{\varrho\in Z(\chi_{(k)})} D(\sigma+ik\gamma_0-\varrho) = \sideset{}{^\star}\sum_{\varrho\in Z(\chi_{(k)})} \left[D(\sigma-\beta+i(k\gamma_0-\gamma)) + D(\sigma-1+\beta+i(k\gamma_0-\gamma))\right], \tag{5.4}$$

where $\sum_{\varrho}^{\star} = \frac{1}{2}\sum_{\beta=\frac12} + \sum_{\beta>\frac12}$.

5.2.1. *Isolating the zero $\beta_0+i\gamma_0$.*

We isolate the summand for $k=1$ and $\varrho=\varrho_0$ and use the following inequality [14, (34) page 325] :

$$D(\sigma-\beta_0) + D(\sigma-1+\beta_0) \ge F(\sigma-\beta_0,0) - c_{2,1}(\eta), \tag{5.5}$$

$$\text{where } \ c_{2,1}(\eta) = -\left[1-\kappa\left(\frac{1}{\delta}+\frac{1}{\sigma_0-\eta_0+\delta}\right)\right]h(0)\eta + \left[1+\kappa\left(\frac{1}{\delta^3}+\frac{1}{(\sigma_0-\eta_0+\delta)^3}\right)\right]m_h\eta^3. \tag{5.6}$$

Thus

$$\begin{aligned}\mathcal{E}_2 &\ge a_1\left(F(\sigma-\beta_0,0) - c_{2,1}(\eta)\right)\\ &+\Bigg(a_1 \sideset{}{^\star}\sum_{\substack{k=1\\ \varrho\in Z(\chi)\\ \varrho\ne\beta_0+i\gamma_0}} + \sum_{\substack{0\le k\le n\\ k\ne 1}} a_k \sideset{}{^\star}\sum_{\varrho\in Z(\chi_{(k)})}\Bigg)\left[D(\sigma-\beta+i(k\gamma_0-\gamma)) + D(\sigma-1+\beta+i(k\gamma_0-\gamma))\right].\end{aligned} \tag{5.7}$$

One of the key arguments in this proof consists in reducing the size of the above sums. Note that $D(\sigma-\beta+iy)$ has the same sign as essentially $\dfrac{\sigma-\beta}{(\sigma-\beta)^2+y^2} - \kappa\dfrac{\sigma-\beta+\delta}{(\sigma-\beta+\delta)^2+y^2}$.
In [27, Lemma 2] Stechkin prove that this is positive when $\sigma-\beta>0$ and under certain conditions for κ,δ. This is done in [14] and we recall this result here.

5.2.2. *A positivity argument for the zeros on the left of σ.*

Proposition 5.2. [14, Proposition 4.2] *Let $1/2<\sigma_0<\sigma<1$ and $0<\eta<\eta_0$. Let h be a positive function satisfying* (3.14)*. We define*

$$\kappa_2(t)=\frac{(2\sigma_0-1)h(0)-\frac{m_h\eta_0^2}{2\sigma_0-1}}{(2t+1)h(0)+\left(\frac{1}{t}+\frac{1}{2\sigma_0-1+t}\right)m_h\eta_0^2}, \tag{5.8}$$

$$\kappa_3(t)=\frac{(2\sigma_0-1)h(0)-\frac{m_h\eta_0^2}{2\sigma_0-1}}{\left(\frac{1}{t}+\frac{1+t}{(2\sigma_0-1+t)^2}\right)h(0)+m_h\eta_0^2\left(\frac{1}{t^3}+\frac{1}{(2\sigma_0-1+t)^3}\right)}. \tag{5.9}$$

If $1/2<\beta<\sigma$ and $y>0$, then $D(\sigma-\beta+iy)+D(\sigma-1+\beta+iy)\geq 0$, as long as $0\leq x\leq \min\left(\kappa_2(t),\kappa_3(t)\right)$ and $t\geq\delta$, where δ is the solution in the interval $(0.5,0.75)$ of the equation $\kappa_2(t)=\kappa_3(t)$. We denote κ the corresponding value of κ_2 at δ: $\kappa=\kappa_2(\delta)=\kappa_3(\delta)$.

Note that as σ_0 and η_0 depend on r, it follows that κ and δ depend on h and r.
This proposition is key in the proof: it allows to reduce the final constant in the zero-free region by a factor of $(1-\kappa)$. As a consequence of Proposition 5.2, we can discard all zeros to the right of σ, so that

$$\begin{aligned}\mathcal{E}_2\geq a_1\left(F(\sigma-\beta_0,0)-c_{2,1}(\eta)\right)\\ +\sum_{k=0}^{n_0}a_k\sideset{}{^\star}\sum_{\substack{\varrho\in Z(\chi_{(k)})\\ \beta\geq\sigma}}\left[D(\sigma-\beta+i(k\gamma_0-\gamma))+D(\sigma-1+\beta+i(k\gamma_0-\gamma))\right].\end{aligned} \tag{5.10}$$

5.2.3. *Estimating the contribution of the zeros on the right of σ.* In previous classical proofs of zero-free regions, the parameter σ was greater than 1 as it was in the region of convergence of $\zeta(s)$. Here, the explicit formula for the smoothed version of $\mathfrak{Re}\frac{L'}{L}$ allows us to choose our parameter $\sigma+i\gamma_0$ inside the critical strip, and thus closer to the zero ϱ_0 we need to locate. This appears in $K(\omega)=\int_0^d\left(a_1e^{-t}-a_0\right)h(t)e^{\omega t}dt$ which needs to be as large as possible to reduce our final constant in the zero-free region. This is feasible as we allow $\omega=\frac{1-\sigma}{1-\beta_0}$ to be positive. On the other hand, a new contribution from the zeros in the vertical strip between σ and 1 arises in the sum over the zeros. We prove here that this one is indeed negligible. From the zero-free region of Theorem 2.2, $\beta\geq\sigma$ implies $|\gamma|\geq k\gamma_0+t_0$. Denoting

$$\mathcal{E}_{2,k}=\sideset{}{^\star}\sum_{\substack{\varrho\in Z(\chi_{(k)})\\ |\gamma|\geq k\gamma_0+t_0}}\left[D(\sigma-\beta+i(k\gamma_0-\gamma))+D(\sigma-1+\beta+i(k\gamma_0-\gamma))\right], \tag{5.11}$$

the inequality (5.10) becomes

$$\mathcal{E}_2\geq a_1\left(F(\sigma-\beta_0,0)-c_{2,1}(\eta)\right)+\sum_{k=0}^{n_0}a_k\mathcal{E}_{2,k}. \tag{5.12}$$

We now use the estimate (3.26) for $D(s)$ to rewrite $\mathcal{E}_{2,k}$ as

$$\mathcal{E}_{2,k}=\sideset{}{^\star}\sum_{\substack{\varrho\in Z(\chi_{(k)})\\ |\gamma|\geq k\gamma_0+t_0}}\left(\eta h(0)\mathcal{E}_{2,k,1}(\varrho)+\mathcal{E}_{2,k,2}(\varrho)\right), \tag{5.13}$$

$$\begin{aligned}\text{with}\quad \mathcal{E}_{2,k,1}(\varrho) =& \frac{\sigma-\beta}{(\sigma-\beta)^2+(k\gamma_0-\gamma)^2}+\frac{\sigma-1+\beta}{(\sigma-1+\beta)^2+(k\gamma_0-\gamma)^2}\\ &-\kappa\left(\frac{\sigma-\beta+\delta}{(\sigma-\beta+\delta)^2+(k\gamma_0-\gamma)^2}+\frac{\sigma-1+\beta+\delta}{(\sigma-1+\beta+\delta)^2+(k\gamma_0-\gamma)^2}\right),\\ \mathcal{E}_{2,k,2}(\varrho) =& H(\sigma-\beta,k\gamma_0-\gamma)+H(\sigma-1+\beta,k\gamma_0-\gamma)\\ &-\kappa\left(H(\sigma-\beta+\delta,k\gamma_0-\gamma)+H(\sigma-1+\beta+\delta,k\gamma_0-\gamma)\right).\end{aligned}$$

The inequalities

$$\begin{aligned}-\eta \leq& \sigma-\beta,\\ 1-2\eta \leq& \sigma-1+\beta+\delta\leq 1,\\ &\sigma-\beta+\delta\leq\delta,\\ &\sigma-1+\beta+\delta\leq\delta+1,\end{aligned}$$

and $|k\gamma_0-\gamma|\geq t_0$ imply

$$\mathcal{E}_{2,k,1}(\varrho)\geq\frac{1}{(k\gamma_0-\gamma)^2}\left(-\eta+\frac{1-2\eta}{t_0^{-2}+1}-\kappa(2\delta+1)\right).$$

By Condition (3.11), $\frac{1}{1+t_0^{-2}}-\kappa(2\delta+1)>0$. Thus

$$\mathcal{E}_{2,k,1}(\varrho)\geq-\frac{\eta}{(k\gamma_0-\gamma)^2}\left(1+\frac{2}{t_0^{-2}+1}\right)=-\frac{\eta}{(k\gamma_0-\gamma)^2}\frac{3+t_0^{-2}}{1+t_0^{-2}}. \tag{5.14}$$

We use (3.19) and (3.20) to respectively bound $|H(\sigma-\beta,k\gamma_0-\gamma)|$ and $|H(x,k\gamma_0-\gamma)|$, with $x=\sigma-1+\beta,\sigma-\beta+\delta$, or $\sigma-1+\beta+\delta$. Since $-\frac{r}{R}\leq-\omega\leq\frac{\sigma-\beta}{\eta}$ and $x\geq\delta-2\eta_0$, we obtain

$$|\mathcal{E}_{2,k,2}(\varrho)|\leq\left(M(-r/R)\eta^2+\frac{1+2\kappa}{\delta-2\eta_0}m_h\eta^3\right)\frac{1}{(\gamma-k\gamma_0)^2}. \tag{5.15}$$

Together with (5.13), (5.14), and (5.15), we deduce for $\mathcal{E}_{2,k}$:

$$\mathcal{E}_{2,k}\geq\left[-\left(\frac{3+t_0^{-2}}{1+t_0^{-2}}h(0)+M(-r/R)\right)-\frac{1+2\kappa}{\delta-2\eta_0}m_h\eta\right]\eta^2\Sigma(k\gamma_0,t_0,\chi_{(k)}), \tag{5.16}$$

$$\text{with}\quad\Sigma(t,t_0,\chi)=\sum_{\substack{\varrho\in Z(\chi)\\ |\gamma|\geq t+t_0}}\frac{1}{(\gamma-t)^2}. \tag{5.17}$$

It remains to establish an upper bound for $\Sigma(t,t_0,\chi)$ when $t\geq 0$. In order to do this, we require explicit bounds for $N(T,\chi)$. Set $\Phi(y)=\dfrac{1}{(y-t)^2}$ and $\Phi'(y)=\dfrac{-2}{(y-t)^3}$. By Stieltjes integration and Theorem 2.3, we have

$$\Sigma(t,t_0,\chi)\leq-\Phi(t+t_0)N_2(t+t_0,q)-\int_{t+t_0}^{\infty}\Phi'(y)N_1(y,q)dy.$$

It remains to bound the last integral.

Lemma 5.3. *Let $t\geq 0$ and t_0 be a positive integer. Let χ be a primitive non-principal character modulo q. Then*

$$\Sigma(0,t_0,\chi)\leq(\log q)\left(\frac{1}{\pi t_0}+\frac{2C_1}{t_0^2}\right)+\frac{\log\frac{et_0}{2\pi}}{\pi t_0}+\frac{4C_1\log t_0+C_1+4C_2}{2t_0^2},$$

and when $t > 0$,

$$\Sigma(t, t_0, \chi) \leq \frac{1}{\pi t_0} \log \frac{q(t+t_0)}{2\pi} + \frac{2C_1}{t_0^2} \log(q(t+t_0)) + \frac{\log\left(1+\frac{t}{t_0}\right)}{\pi t} + \frac{C_1}{t t_0}\left(1 - \frac{t_0}{t}\log\left(1+\frac{t}{t_0}\right)\right) + \frac{2C_2}{t_0^2}.$$

To bound $\eta\Sigma(0, t_0, \chi_k)$, we use $\frac{1}{r\log(q\gamma_0)} \leq \frac{1}{r\log Q_0}$ and $\frac{\log q}{r\log(q\gamma_0)} \leq \frac{\log q_1}{r\log Q_0}$, so that

$$\eta\Sigma(0, t_0, \chi) \leq \frac{(\log q_1)\left(\frac{1}{\pi t_0} + \frac{2C_1}{t_0^2}\right) + \frac{\log\frac{e t_0}{2\pi}}{\pi t_0} + \frac{4C_1\log t_0 + C_1 + 4C_2}{2t_0^2}}{r\log Q_0}.$$

To bound $\eta\Sigma(k\gamma_0, t_0, \chi_k)$ when $k \geq 1$, we also use $\eta\log\frac{q(k\gamma_0+t_0)}{2\pi} = \frac{\log\frac{q(k\gamma_0+t_0)}{2\pi}}{r\log(q\gamma_0)} \leq \frac{\log\frac{Q_0\left(k+\frac{t_0}{H_q}\right)}{2\pi}}{r\log Q_0}$.
In addition, we note that $\frac{\log\left(1+\frac{t}{t_0}\right)}{\pi t} + \frac{C_1}{t t_0}\left(1-\frac{t_0}{t}\log\left(1+\frac{t}{t_0}\right)\right)$ is positive and decreases with t.

$$\text{Thus}\quad \eta\Sigma(k\gamma_0, t_0, \chi_k) \leq \Big[\frac{1}{\pi t_0}\log\left(\frac{Q_0}{2\pi}\left(k+\frac{t_0}{H_q}\right)\right) + \frac{2C_1}{t_0^2}\log\left(Q_0\left(k+\frac{t_0}{H_q}\right)\right)$$
$$+ \frac{1}{\pi k H_q}\log\left(1+\frac{kH_q}{t_0}\right) + \frac{C_1}{t_0 k H_q}\left(1-\frac{t_0}{kH_q}\log\left(1+\frac{kH_q}{t_0}\right)\right) + \frac{2C_2}{t_0^2}\Big]\frac{1}{r\log Q_0}. \quad (5.18)$$

We rearrange the terms and obtain $\eta\Sigma(k\gamma_0, t_0, \chi_{(k)}) \leq s_k$, with

$$s_0 = \Big(\frac{\log\left(\frac{e t_0 q_1}{2\pi}\right)}{\pi t_0} + \frac{4C_1\log(q_1 t_0) + C_1 + 4C_2}{2t_0^2}\Big)\eta_0, \quad (5.19)$$

$$s_k = \Big(\Big(\frac{1}{\pi t_0} + \frac{2C_1}{t_0^2}\Big)\log\Big(Q_0\Big(k+\frac{t_0}{H_q}\Big)\Big) - \frac{\log(2\pi)}{\pi t_0} + \frac{2C_2}{t_0^2} + \frac{\left(\frac{1}{\pi} - \frac{C_1}{kH_q}\right)\log\left(1+\frac{kH_q}{t_0}\right)}{kH_q} + \frac{C_1}{t_0 k H_q}\Big)\eta_0, \quad (5.20)$$

for $k \geq 1$ This allows to rewrite the bound (5.16) for $\mathcal{E}_{2,k}$ as $\mathcal{E}_{2,k} \geq -c_{2,2}(k,\eta)$, with

$$c_{2,2}(k,\eta) = \left(\left(\frac{3+t_0^{-2}}{1+t_0^{-2}}h(0) + M(-r/R)\right)\eta + \frac{1+2\kappa}{\delta - 2\eta_0}m_h\eta^2\right)s_k. \quad (5.21)$$

This gives a final bound for $\mathcal{E}_2$ and (5.12) becomes

Lemma 5.4. *Let* $\sigma, \gamma_0, \kappa, \delta, t_0$ *be as in Section 3.1 and Proposition 5.2. Then*

$$-\mathcal{E}_2 \leq -a_1 F(\sigma - \beta_0, 0) + \mathcal{C}_2(\eta), \quad (5.22)$$

$$\textit{with}\quad \mathcal{C}_2(\eta) = a_1 c_{2,1}(\eta) + \sum_{k=0}^{n_0} a_k c_{2,2}(k,\eta), \quad (5.23)$$

where $c_{2,1}$ *and* $c_{2,2}$ *are defined in* (5.6) *and* (5.21) *respectively.*

Remark. Note that Lehman's method as used in [21] provides a larger bound with $\frac{2}{t_0^2}$ instead of $\frac{2C_1}{t_0^2}$ in the main factor $\log(q(t+t_0))$.

5.3. **Study of the Gamma-term $\mathcal{E}_3$.**

Lemma 5.5. *Let $\sigma, \gamma_0, \kappa, \delta, t_0$ be as in Section 3.1 and Proposition 5.2. Then*

$$\mathcal{E}_3 \leq \frac{(1-\kappa)Ah(0)}{2}\eta\log(q\gamma_0) + \mathcal{C}_3(\eta), \tag{5.24}$$

with $\mathcal{C}_3(\eta) = \eta h(0)\Bigg[a_0\left(-\frac{1-\kappa}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}(3/2) - \frac{\kappa}{2}\frac{\Gamma'}{\Gamma}((\sigma_0+2+\delta)/2)\right)$

$$+\sum_{k=1}^{n_0} a_k\Big(\frac{1-\kappa}{2}\log\frac{k}{2\pi} + \frac{\arctan\left(\frac{kH_q}{\sigma_0}\right)+\kappa\arctan\left(\frac{kH_q}{\sigma_0+\delta}\right)}{2kH_q} + \frac{4+\kappa(2+\delta)^2}{4(kH_q)^2}\Big)\Bigg]. \tag{5.25}$$

Proof. $\frac{\Gamma'}{\Gamma}$ is an increasing function of the real variable, so for $k=0$ and $\sigma_0 < \sigma < 1$:

$$\mathfrak{G}_{\kappa,\delta}(\sigma+2) = \frac{\Gamma'}{\Gamma}\left(\frac{\sigma+2}{2}\right) - \kappa\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+2+\delta}{2}\right) \leq \frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right) - \kappa\frac{\Gamma'}{\Gamma}\left(\frac{\sigma_0+2+\delta}{2}\right). \tag{5.26}$$

For $k \geq 1$, the identity from [19, page 12] gives

$$\frac{\Gamma'}{\Gamma}\left(\frac{x+iy}{2}\right) = \frac{1}{2}\log\frac{x^2+y^2}{4} - \frac{x}{x^2+y^2} + \mathfrak{Re}\int_0^{+\infty}\frac{u-[u]-\frac{1}{2}}{\left(u+\frac{x+iy}{2}\right)^2}du. \tag{5.27}$$

We have $\left|\mathfrak{Re}\int_0^{+\infty}\frac{u-[u]-\frac{1}{2}}{(u+\frac{x+iy}{2})^2}du\right| \leq \frac{1}{y}\arctan\left(\frac{y}{x}\right)$, so isolating $\log\frac{|y|}{2}$ in (5.27) gives the estimate

$$\left|\frac{\Gamma'}{\Gamma}\left(\frac{x+iy}{2}\right) - \left(\log\frac{|y|}{2} - \frac{x}{x^2+y^2}\right)\right| \leq \frac{1}{y}\arctan\left(\frac{y}{x}\right) + \frac{x^2}{2y^2}, \tag{5.28}$$

and then $|\mathfrak{G}_{\kappa,\delta}(\sigma+ik\gamma_0+\mathfrak{a})| \leq \left|\frac{1-\kappa}{2}\log\frac{k\gamma_0}{2} - \frac{\sigma+\mathfrak{a}}{(\sigma+\mathfrak{a})^2+(k\gamma_0)^2} + \kappa\frac{\sigma+\mathfrak{a}+\delta}{(\sigma+\mathfrak{a}+\delta)^2+(k\gamma_0)^2}\right|$

$$+\frac{\arctan\left(\frac{k\gamma_0}{\sigma+\mathfrak{a}}\right)+\kappa\arctan\left(\frac{k\gamma_0}{\sigma+\mathfrak{a}+\delta}\right)}{k\gamma_0} + \frac{(\sigma+\mathfrak{a})^2+\kappa(\sigma+\mathfrak{a}+\delta)^2}{2(k\gamma_0)^2}. \tag{5.29}$$

Both $\frac{x}{x^2+y^2} - \kappa\frac{x+\delta}{(x+\delta)^2+y^2}$ and $\frac{1}{y}\arctan\frac{y}{x}$ are nonnegative and decreasing with y when $\kappa < \frac{x}{x+\delta}$ (which is the case from (3.10) since $\kappa < \frac{\sigma_0}{1+\delta}$). Since $\gamma_0 \geq H_q$ and $\sigma_0 < \sigma < 1$, then

$$|\mathfrak{G}_{\kappa,\delta}(\sigma+ik\gamma_0+\mathfrak{a})| \leq (1-\kappa)\log\frac{k\gamma_0}{2} + \frac{\arctan\left(\frac{kH_q}{\sigma_0}\right)+\kappa\arctan\left(\frac{kH_q}{\sigma_0+\delta}\right)}{kH_q} + \frac{4+\kappa(2+\delta)^2}{2(kH_q)^2}. \tag{5.30}$$

We conclude the proof by combining (3.39) with (5.26) and (5.30). □

5.4. **Study of the error term with the Gamma integral $\mathcal{E}_4$.** First, we establish a preliminary result to estimate $\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)$:

Lemma 5.6. *Let $\mathfrak{a} = 0$ or 1 and T a real number. Let $u_\mathfrak{a}$ be the root of* $\log\frac{u}{2} - \frac{\mathfrak{a}+1/2}{(\mathfrak{a}+1/2)^2+u^2}$,

$$b_\mathfrak{a} = \gamma + 3\log 2 + (-1)^\mathfrak{a}\frac{\pi}{2},$$

$$c_\mathfrak{a} = -\log 2 - \frac{\mathfrak{a}+1/2}{(\mathfrak{a}+1/2)^2+u_\mathfrak{a}^2} + \frac{1}{u_\mathfrak{a}}\arctan\left(\frac{u_\mathfrak{a}}{\mathfrak{a}+1/2}\right) + \frac{(\mathfrak{a}+1/2)^2}{2u_\mathfrak{a}^2}.$$

Then

$$\left|\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{\mathfrak{a}+1/2+iT}{2}\right)\right| \leq U_{\mathfrak{a}}(T) = \begin{cases} b_{\mathfrak{a}} & \text{if } \ |T| \leq u_{\mathfrak{a}}, \\ \log|T| + c_{\mathfrak{a}} & \text{otherwise.} \end{cases}$$

Numerically, we have

$\mathfrak{a}$	$u_{\mathfrak{a}}$	$b_{\mathfrak{a}}$	$c_{\mathfrak{a}}$
0	$2.2054\ldots$	$4.2274\ldots$	$-0.1540\ldots$
1	$2.4093\ldots$	$1.0858\ldots$	$-0.2647\ldots$

Proof. Thanks to Stirling's formula $\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{x+iy}{2}\right) = -\gamma - \frac{2x}{x^2+y^2} + \sum_{n=1}^{+\infty}\left(\frac{1}{n} - \frac{2(2n+x)}{(2n+x)^2+y^2}\right)$, we see that $\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{\mathfrak{a}+1/2+iT}{2}\right)$ increases with $|T|$. Thus when $|T| \leq u_{\mathfrak{a}}$,

$$\left|\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{\mathfrak{a}+1/2+iT}{2}\right)\right| \leq \left|\frac{\Gamma'}{\Gamma}\left(\frac{\mathfrak{a}+1/2}{2}\right)\right| = \gamma + 3\log 2 + (-1)^{\mathfrak{a}}\frac{\pi}{2}.$$

When $|T| \geq u_{\mathfrak{a}}$, we apply the inequality (5.28) together with the fact that $\log\frac{|T|}{2} - \frac{\mathfrak{a}+1/2}{(\mathfrak{a}+1/2)^2+T^2}$ is nonnegative. Bounding $-\frac{\mathfrak{a}+1/2}{(\mathfrak{a}+1/2)^2+T^2} + \frac{1}{|T|}\arctan\left(\frac{|T|}{\mathfrak{a}+1/2}\right) + \frac{(\mathfrak{a}+1/2)^2}{2T^2}$ with its value at $|T| = u_{\mathfrak{a}}$, it follows

$$\left|\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{\mathfrak{a}+1/2+iT}{2}\right)\right| \leq \log\frac{|T|}{2} - \frac{\mathfrak{a}+1/2}{(\mathfrak{a}+1/2)^2+u_{\mathfrak{a}}^2} + \frac{\arctan\left(\frac{u_{\mathfrak{a}}}{\mathfrak{a}+1/2}\right)}{u_{\mathfrak{a}}} + \frac{(\mathfrak{a}+1/2)^2}{2u_{\mathfrak{a}}^2}.$$

□

Let $I_{\mathfrak{a}}$ be the integral

$$I_{\mathfrak{a}}(x,y) = \int_0^{+\infty} \frac{U_{\mathfrak{a}}(t)}{x(x^2+(y-t)^2)}dt. \tag{5.31}$$

Applying (3.20) to bound the H-terms in definition (3.40), it follows for $\sigma > \sigma_0$ that

$$\begin{aligned}\mathcal{E}_4 \leq a_0 m_h \eta^3 &\left(\frac{1}{\sigma_0^3} + \frac{\kappa}{(\sigma_0+\delta)^3} + \frac{I_0(\sigma_0-1/2,0)+\kappa I_0(\sigma_0+\delta-1/2,0)}{\pi}\right) \\ &+ \frac{m_h\eta^3}{2\pi}\sum_{k=1}^{n_0} a_k\big[I_{\mathfrak{a}}(\sigma_0-1/2,k\gamma_0) + I_{\mathfrak{a}}(\sigma_0-1/2,-k\gamma_0) \\ &\qquad + \kappa\left(I_{\mathfrak{a}}(\sigma_0+\delta-1/2,k\gamma_0) + I_{\mathfrak{a}}(\sigma_0+\delta-1/2,-k\gamma_0)\right)\big].\end{aligned} \tag{5.32}$$

Note that $I_{\mathfrak{a}}(x,y)$ is decreasing with x. The following lemma provides bounds for $I_{\mathfrak{a}}$.

Lemma 5.7. *Let* $\mathfrak{a} = 0$ *or* 1, $x > 0, y > u_{\mathfrak{a}}+1$. *Then*

$$I_0(x,0) \leq J_0(x), \tag{5.33}$$

$$I_{\mathfrak{a}}(x,-y) \leq J_{\mathfrak{a},1}(x,y), \tag{5.34}$$

$$I_{\mathfrak{a}}(x,y) \leq J_{\mathfrak{a},2}(x,y) + J_{\mathfrak{a},3}(x,y) + (\log y)J_4(x,y), \tag{5.35}$$

$$\text{with}\quad J_0(x) = \frac{(b_0-c_0)\arctan\frac{u_0}{x}+\frac{c_0\pi}{2}}{x^2}+\frac{\log u_0+1}{xu_0},$$
$$J_{\mathfrak{a},1}(x,y) = \frac{(b_{\mathfrak{a}}-c_{\mathfrak{a}})\arctan\frac{y+u_{\mathfrak{a}}}{x}-b_{\mathfrak{a}}\arctan\frac{y}{x}+\frac{\pi c_{\mathfrak{a}}}{2}}{x^2}-\frac{\log(u_{\mathfrak{a}}+y)}{xy}+\frac{u_{\mathfrak{a}}\log u_{\mathfrak{a}}}{xy(u_{\mathfrak{a}}+y)},$$
$$J_{\mathfrak{a},2}(x,y) = \frac{b_{\mathfrak{a}}\left(\arctan\frac{y}{x}-\arctan\frac{y-u_{\mathfrak{a}}}{x}\right)}{x^2},$$
$$J_{\mathfrak{a},3}(x,y) = \frac{c_{\mathfrak{a}}\left(\frac{\pi}{2}+\arctan\frac{y-u_{\mathfrak{a}}}{x}\right)}{x^2},$$
$$J_4(x,y) = \frac{\pi}{x^2}+\frac{\pi}{x^2(\log y)^{3/2}}+\frac{2\sqrt{\log y}}{xy}+\frac{2}{xy(\log y)^{3/2}}.$$

Proof. Let $t\geq u_0$. We get (5.33) from the fact that $\frac{\log t}{x(x^2+t^2)}\leq\frac{\log t}{xt^2}$, and thus

$$I_0(x,0)\leq\int_0^{u_0}\frac{b_0}{x(x^2+t^2)}dt+\int_{u_0}^{+\infty}\frac{\log t}{xt^2}dt+\int_{u_0}^{+\infty}\frac{c_0}{x(x^2+t^2)}dt.$$

Let $t>u_{\mathfrak{a}}$. The announced inequality (5.34) follows from $\frac{\log t}{x(x^2+(y+t)^2)}\leq\frac{\log t}{x(y+t)^2}$ and from

$$I_{\mathfrak{a}}(x,-y)\leq\int_0^{u_{\mathfrak{a}}}\frac{b_{\mathfrak{a}}}{x(x^2+(y+t)^2)}dt+\int_{u_{\mathfrak{a}}}^{+\infty}\frac{\log t}{x(y+t)^2}+\frac{c_{\mathfrak{a}}}{x(x^2+(y+t)^2)}dt.$$

Let $\epsilon>0$ be a parameter (depending on y), and split the integral $I_{\mathfrak{a}}(x,y)$ at $t=u_{\mathfrak{a}}$. The first integral is exactly $J_{\mathfrak{a},2}(x,y)$:

$$\int_0^{u_{\mathfrak{a}}}\frac{U_{\mathfrak{a}}(t)}{x(x^2+(y-t)^2)}dt=\frac{b_{\mathfrak{a}}}{x^2}\left(\arctan\frac{y}{x}-\arctan\frac{y-u_{\mathfrak{a}}}{x}\right)=J_{\mathfrak{a},2}(x,y). \tag{5.36}$$

Now for $t>u_{\mathfrak{a}}$, Lemma 5.6 gives $U_{\mathfrak{a}}(t)=\log t+c_{\mathfrak{a}}$. It is immediate that

$$\int_{u_{\mathfrak{a}}}^{+\infty}\frac{c_{\mathfrak{a}}}{x(x^2+(y-t)^2)}dt=\frac{c_{\mathfrak{a}}}{x^2}\left(\frac{\pi}{2}+\arctan\frac{y-u_{\mathfrak{a}}}{x}\right)=J_{\mathfrak{a},3}(x,y). \tag{5.37}$$

To compute the $(\log t)$-part, we split at $t=y(1-\epsilon)$ and $y(1+\epsilon)$.
When $u_{\mathfrak{a}}<t<y(1-\epsilon)$ or $t>y(1-\epsilon)$, we use $\frac{\log t}{x^2+(y-t)^2}\leq\frac{\log t}{(y-t)^2}$.
Assuming $y>u_{\mathfrak{a}}+1$, we obtain

$$\begin{aligned}&\left(\int_{u_{\mathfrak{a}}}^{y(1-\epsilon)}+\int_{y(1+\epsilon)}^{+\infty}\right)\frac{\log t}{x(y-t)^2}dt\\&=\frac{1}{xy\epsilon}\Big(-\epsilon\log(y-u_{\mathfrak{a}})+\frac{\epsilon u_{\mathfrak{a}}\log u_{\mathfrak{a}}}{u_{\mathfrak{a}}-y}+2\log y+(1-\epsilon)\log(1-\epsilon)+(1+\epsilon)\log(1+\epsilon)\Big)\\&\leq\frac{2\left(\log y+\epsilon^2\right)}{xy\epsilon}.\end{aligned}$$

When $y(1-\epsilon)<t<y(1+\epsilon)$, we have

$$\int_{y(1-\epsilon)}^{y(1+\epsilon)}\frac{\log t}{x(x^2+(y-t)^2)}dt\leq\int_{y(1-\epsilon)}^{y(1+\epsilon)}\frac{\log(y(1+\epsilon))}{x(x^2+(y-t)^2)}dt=\frac{2\log(y(1+\epsilon))\arctan\frac{y\epsilon}{x}}{x^2}\leq\frac{\pi(\log y+\epsilon)}{x^2}.$$

Choosing $\epsilon=\frac{1}{\sqrt{\log y}}$ and adding the two last inequalities, we obtain

$$\int_{u_{\mathfrak{a}}}^{+\infty}\frac{\log t}{x(x^2+(y-t)^2)}dt\leq(\log y)J_4(x,y). \tag{5.38}$$

(5.35) follows from (5.36), (5.37) and (5.38). □

We combine the bound (5.32) for $\mathcal{E}_4$ with Lemma 5.7. In particular, to bound $\eta(I_{\mathfrak{a}}(\sigma_0-1/2,k\gamma_0)+\kappa I_{\mathfrak{a}}(\sigma_0+\delta-1/2,k\gamma_0))$, we use the fact that $\eta\log(k\gamma_0)=\frac{1+\frac{\log k}{\log\gamma_0}}{r(1+\frac{\log q}{\log\gamma_0})}\le\frac{\log(kH_q)}{r\log H_q}$.

Lemma 5.8. *Let $\sigma,\gamma_0,\kappa,\delta,t_0$ be as in Section 3.1 and Proposition 5.2. Then*

$$\mathcal{E}_4\le\mathcal{C}_4(\eta), \tag{5.39}$$

with

$$\mathcal{C}_4(\eta)=\eta^3a_0m_h\left(\frac{1}{\sigma_0^3}+\kappa\frac{1}{(\sigma_0+\delta)^3}+\frac{J_0(\sigma_0-1/2)+\kappa J_0(\sigma_0-1/2+\delta)}{\pi}\right)$$
$$+\eta^3\frac{m_h}{2\pi}\sum_{k=1}^{n_0}\sum_{j=1}^{3}a_k\max_{\mathfrak{a}=0,1}\left(J_{\mathfrak{a},j}(\sigma_0-1/2,kH_q)+\kappa J_{\mathfrak{a},j}(\sigma_0-1/2+\delta,kH_q)\right)$$
$$+\eta^2\frac{m_h}{2r\pi}\sum_{k=1}^{n_0}a_k\left(J_4(\sigma_0-1/2,kH_q)+\kappa J_4(\sigma_0-1/2+\delta,kH_q)\right). \tag{5.40}$$

5.5. **Choice of the smooth weight h.** Let $\lambda>0$ and c_1,c_2,x_1,x_2 some fixed real numbers. Consider

$$g(x)=c_1\cos(x_1x)+c_2\cosh(x_2x)-(c_1\cos(x_1\lambda)+c_2\cosh(x_2\lambda)) \tag{5.41}$$

for $-\lambda\le x\le\lambda$ and $g(x)=0$ otherwise. We have $g''(x)=-c_1\cos(x_1u)x_1^2+c_2\cosh(x_2u)x_2^2$. Note that g is compactly supported, non-negative, even, infinitely differentiable, and that g'' is even, negative, and increasing on $(0,\lambda)$. Let $d_5=2\lambda$, and define h the self-convolution of g:

$$h(u)=(g\star g)(u)=\int_{u-\lambda}^{\lambda}g(x)g(u-x)dx,$$

when $0<u<d_5$, and $h(u)=0$ otherwise. Note that h satisfies Conditions (3.14). We have $h''(u)=\displaystyle\int_{-\infty}^{\infty}g(x)g''(u-x)dx$ and denote $m_h=\max_{0<u<d_5}|h''(u)|$. Numerically, we observe that $m_h=-h''(0)$.

Remark. In his famous article about the least prime in an arithmetic progression [9, Lemmas 7.1-7.4], Heath-Brown used various smoothed versions of $-\frac{L'}{L}(s,\chi)$. Later Xylouris followed one of Heath-Brown's remarks and proposed some other weights. For instance [14] used the weight as defined in [9, Lemma 7.4]. In [12] Jan and Kwon compared all the weights proposed by Heath-Brown and Xylouris which allowed them to improve the last zero-free region for zeta (by 1.322%). After investigating all these five weights, we find that the weight given in [30, page 75] by Xylouris provides here the best constant for the zero-free region.

5.6. **Choice of the trigonometric polynomial $P(x)$.** The last area where we seek some final improvement concerns the trigonometric polynomial. We take here the opportunity to briefly describe what we know about the progresses concerning this aspect. For the Riemann zeta function, de la Vallée Poussin used the following quadratic polynomial: $2(1+\cos x)^2=3+4\cos x+\cos(2x)$, and Landau showed that the value for the constant R_0 was given by

$$\frac{A}{2\left(\sqrt{a_1}-\sqrt{a_0}\right)^2}, \tag{5.42}$$

with $A=\sum_{k=1}^{n_0}a_k$. In the case of de la Vallée Poussin's quadratic polynomial, (5.42) equals $34.82050\ldots$. Finding a polynomial satisfying (3.3) and such that the value for (5.42) is as small as possible becomes challenging as the degree of the polynomial increases. In analysis, this problem is referred to as Landau's extremal problems (see [25, Section 9]). In 1970, Stechkin [27] introduced the degree 4 polynomial:

$$P(x) = 8(0.9126 + \cos x)^2(0.2766 + \cos x)^2$$
$$= 11.18... + 19.07... \cos x + 11.67... \cos(2x) + 4.75... \cos(3x) + \cos(4x), \quad (5.43)$$

which brings (5.42) down to 17.42622 . . .. Together with his clever idea mentioned in Section 5.2.2, he was able to reduce the constant in the zero-free region to 9.65. In 1975, minor modifications of Stechkin's idea led Rosser and Schoenfeld [26] to compute 9.645908801 instead, and in 1984, McCurley [19] generalized this to Dirichlet L-functions. In earlier work [14] and in [13], we used a minor modification of Stechkin's polynomial (5.43):

$$P(x) = 8(0.91 + \cos x)^2(0.265 + \cos x)^2$$
$$= 10.91... + 18.63... \cos x + 11.45... \cos(2x) + 4.7 \cos(3x) + \cos(4x). \quad (5.44)$$

We reduced the constant for zeta to 5.69693 [14], which was the first significant result 35 years after Stechkin, as well as the constant for Dirichlet L-functions of McCurley's [19], obtaining the constant 6.4355 [13]. In 2012, I first heard through Olivier Ramaré about a forgotten article of Kondratev: in [18] he found a polynomial of degree 8 which lead to a constant 17.27230 . . . for (5.42). Consequently, it reduced Stechkin's constant for zeta's zero-free region to 9.54789695. In 2014, Mossinghoff and Trudgian [21] used the author's method and also investigated numerically other trigonometric polynomials of higher degrees (up to 40). Among these, they find the following polynomial of degree 16 to produce the smallest constant for zeta's zero-free region:

TABLE 1. $P(x) = \left|\sum_{k=0}^{16} c_k e^{ikx}\right|^2 = \sum_{k=0}^{16} a_k \cos(kx)$

c_0	1	a_0	1
c_1	-2.09100370089199	a_1	1.74126664022806
c_2	0.414661861733616	a_2	1.128282822804652
c_3	4.94973437766435	a_3	0.5065272432186642
c_4	2.26052224951171	a_4	0.1253566902628852
c_5	8.58599241204357	a_5	$9.35696526707405 \cdot 10^{-13}$
c_6	6.87053689828658	a_6	$4.546614790384321 \cdot 10^{-13}$
c_7	22.6412990090005	a_7	0.01201214561729989
c_8	6.76222005424994	a_8	0.006875849760911001
c_9	50.2233943767588	a_9	$7.77030543093611 \cdot 10^{-12}$
c_{10}	8.07550113395201	a_{10}	$2.846662294985367 \cdot 10^{-7}$
c_{11}	223.771572768515	a_{11}	0.001608306592372963
c_{12}	487.278135806977	a_{12}	0.001017994683287104
c_{13}	597.268928658734	a_{13}	$2.838909054508971 \cdot 10^{-7}$
c_{14}	473.937203439807	a_{14}	$5.482482041999887 \cdot 10^{-6}$
c_{15}	237.271715181426	a_{15}	$2.412958794855076 \cdot 10^{-4}$
c_{16}	59.6961898512813	a_{16}	$1.281001290654868 \cdot 10^{-4}$
		A	3.523323140225021

We note that this polynomial is close to best possible: with it (5.42) equals 17.24998 . . . while the optimal value for (5.42) over all even trigonometric polynomial with non-negative coefficients and satisfying $a_1 > a_0$ is no smaller than 17.23415 (as proven in [21, Theorem 2]).

5.7. **Computations.** We use the polynomial given in Section 5.6 and the smooth function h depending on $c_1, c_2, x_1, x_2, \lambda$ as defined in Section 5.5. We set $c_1 = 1$ and $x_1 = 1$ and for each set values for c_2, x_2, and λ, we choose r and η_1 so as to make the zero-free region constant R_0 as given by (4.12) as small as possible. We repeat the step with replacing the value of R at step k by the value of R_0 at step $k+1$. We repeat until we get the first two decimal digits between R and r to be the same. We record our results in the following tables and display up to the first four decimal digits for each value computed (that is for κ, δ, and R_0).

TABLE 2. For $3 \leq q \leq 1\,000$

Step	R	r	c_2	x_2	λ	η_1	κ	δ	R_0
1	9.6459	5.857	1.361	0.765	0.551	0.0007	0.4414	0.6198	5.8579
2	5.8579	5.622	1.875	0.639	0.533	0.0011	0.4381	0.6208	5.6223
3	5.6223	5.599	0.194	1.650	0.531	0.0012	0.4378	0.6209	5.5992
4	5.5992	5.596	0.111	2.013	0.531	0.0012	0.4377	0.6210	5.5968

TABLE 3. For $1000 < q \leq 400\,000$

Step	R	r	c_2	x_2	λ	η_1	κ	δ	R_0
1	9.6459	5.857	1.384	0.759	0.551	0.0007	0.4414	0.6198	5.8579
2	5.8579	5.622	1.494	0.710	0.533	0.0011	0.4381	0.6208	5.6223
3	5.6223	5.599	0.189	1.665	0.531	0.0012	0.4378	0.6209	5.5992
4	5.5992	5.596	0.086	2.190	0.531	0.0012	0.4377	0.6210	5.5968

6. Some complementary proofs

6.1. **Explicit formulae for Dirichlet L-functions.** We give here an explicit formula relating sums over zeros of a Dirichlet L-function and sums over primes. This corrects a mistake in [14, Theorem 3.1] for the case of primitive characters.

Theorem 6.1. *Let ϕ be a complex valued function such that*

(A) *ϕ is C^1 on $\mathbb{R} - S$, where S is a finite set of points a_i where both ϕ and its derivative ϕ' have at worse removable discontinuities.Moreover, at these points ϕ verifies $\phi(a_i) = \frac{1}{2}[\phi(a_i+0) + \phi(a_i-0)]$.*

(B) *There exists $b > 0$ such that $\phi(x)e^{x/2}$ and $\phi'(x)e^{x/2}$ are $\mathcal{O}(e^{-(\frac{1}{2}+b)|x|})$ as $x \to \infty$.*

For each $a < 1$ verifying $0 < a < b$, ϕ has a Laplace transform $\Phi(s) = \int_0^{+\infty} \phi(x)e^{-sx}dx$ which is holomorphic in $-(1+a) < \sigma < a$ and which is $\mathcal{O}(1/|t|)$ in $-(1+a) \leq \sigma \leq a$. Let $q \in \mathbb{N}$ and χ a primitive character modulo q. Let $\mathfrak{a} = 0$ if $\chi(-1) = 1$, 1 if $\chi(-1) = -1$. Then

$$\sum_{n\geq 1} \Lambda(n)\chi(n)\phi(\log n) = \phi(0)\log\frac{q}{\pi} - \sum_{\varrho\in Z(\chi)} \Phi(-\varrho) - \sum_{n\geq 1} \frac{\Lambda(n)\overline{\chi}(n)\phi(-\log n)}{n} + \frac{1}{2i\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)\Phi(-s)ds, \quad (6.1)$$

and

$$\sum_{n\geq 1} \Lambda(n)\phi(\log n) = \Phi(-1) + \Phi(0) - \phi(0)\log\pi - \sum_{\varrho\in Z(\zeta)} \Phi(-\varrho) - \sum_{n\geq 1} \frac{\Lambda(n)\phi(-\log n)}{n} + \frac{1}{2i\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)\Phi(-s)ds, \quad (6.2)$$

where $Z(\chi)$ and $Z(\zeta)$ are the sets of non-trivial zeros of respectively $\mathcal{L}(s,\chi)$ and $\zeta(s)$.

Proof. The Inverse Laplace transform gives $\phi(\log n) = \frac{1}{2i\pi}\int_{-(1+a)-i\infty}^{-(1+a)+i\infty}\Phi(s)n^s ds \quad (n\geq 1)$. Combining this with a change of variable ($s\to -s$), we obtain

$$\sum_{n\geq 1}\Lambda(n)\chi(n)\phi(\log n) = \frac{1}{2i\pi}\int_{1+a-i\infty}^{1+a+i\infty} -\frac{L'}{L}(s,\chi)\Phi(-s)ds = \mathcal{I}_{1+a-i\infty}^{1+a+i\infty}, \tag{6.3}$$

with $\mathcal{I}_{x+iy}^{x'+iy'} = \frac{1}{2i\pi}\int_{x+iy}^{x'+iy'} -\frac{L'}{L}(s,\chi)\Phi(-s)ds$. We consider $\mathcal{I}_{1+a-iT}^{1+a+iT}$ the truncated integral, where $T>0$ is fixed and does not equal the ordinate of a zero of $L(s,\chi)$. We move the contour of integration to $[-a-iT,-a+iT]$, and collect the residues of $-\frac{L'}{L}(s,\chi)\Phi(-s)$ in the issued rectangle. The poles in this rectangle are simple and located at the non-trivial zeros $\varrho\in Z(\chi)$, each with residue -1. In addition, if χ is even, there is another simple pole at $s=0$, also with residue -1. Letting $\mathcal{P}_{T,\chi}$ be the pole contribution in the rectangle, we have from Cauchy's Residue Theorem that

$$\mathcal{I}_{1+a-iT}^{1+a+iT} = \mathcal{P}_{T,\chi} + \mathcal{I}_{-a-iT}^{-a+iT} + \mathcal{I}_{-a+iT}^{1+a+iT} - \mathcal{I}_{-a-iT}^{1+a-iT}, \tag{6.4}$$

with

$$\mathcal{P}_{T,\chi} = -(1-\mathfrak{a})\Phi(0) - \sum_{\substack{\varrho\in Z(\chi)\\ |\mathfrak{Im}\varrho|<T}}\Phi(-\varrho). \tag{6.5}$$

It follows from Condition (B) that $\mathcal{I}_{-a+iT}^{1+a+iT}$ and $\mathcal{I}_{-a-iT}^{1+a-iT}$ have limit 0 as $T\to\infty$. Moreover, the functional equation [5, (13) (14) page 71]

$$-\frac{L'}{L}(s,\chi) = \log\frac{q}{\pi} + \frac{L'}{L}(1-s,\overline{\chi}) + \frac{1}{2}\left\{\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1-s+\mathfrak{a}}{2}\right)\right\} \tag{6.6}$$

allows to split $\mathcal{I}_{-a-i\infty}^{-a+i\infty}$ as the sum of three integrals:

$$\mathcal{I}_{-a-i\infty}^{-a+i\infty} = I_{\mathfrak{a},1}(\chi) + I_{\mathfrak{a},2}(\chi) + I_{\mathfrak{a},3}(\chi), \tag{6.7}$$

$$\text{with } I_{\mathfrak{a},1}(\chi) = \frac{1}{2i\pi}\int_{-a-i\infty}^{-a+i\infty}\log\frac{q}{\pi}\Phi(-s)ds = \phi(0)\log\frac{q}{\pi}, \tag{6.8}$$

$$I_{\mathfrak{a},2}(\chi) = \frac{1}{2i\pi}\int_{-a-i\infty}^{-a+i\infty}\frac{L'}{L}(1-s,\overline{\chi})\Phi(-s)ds = -\sum_{n\geq 1}\frac{\Lambda(n)\overline{\chi}(n)\phi(-\log n)}{n}, \tag{6.9}$$

$$I_{\mathfrak{a},3}(\chi) = \frac{1}{2i\pi}\int_{-a-i\infty}^{-a+i\infty}\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1-s+\mathfrak{a}}{2}\right)\right)\Phi(-s)ds. \tag{6.10}$$

To evaluate $I_{\mathfrak{a},3}(\chi)$, we move the path of integration to the $\frac{1}{2}$-line on which Γ verifies

$$\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1-s+\mathfrak{a}}{2}\right)\right) = \mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right).$$

Condition (B) and the fact that $\frac{\Gamma'}{\Gamma}(s)$ has a simple pole at $s=0$ with residue -1 lead to

$$I_{\mathfrak{a},3}(\chi) = \frac{1}{2i\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\mathfrak{Re}\frac{\Gamma'}{\Gamma}(s+\mathfrak{a})\Phi(-s)ds + (1-\mathfrak{a})\Phi(0). \tag{6.11}$$

Note that this contribution of $\Phi(0)$ cancels the one arising in (6.5). Finally, together with (6.4), (6.7), (6.8), (6.9), and (6.11), we rewrite (6.3) in the form announced in (6.1).

The formula (6.2) is obtained similarly. We recall that the poles of $-\frac{\zeta'}{\zeta}(s)$ are all simple and are located at $s=1$, with residue 1, and at the non-trivial zeros $\varrho\in Z(\zeta)$, with residue -1. In this case the polar contribution is $\mathcal{P}_T = \Phi(-1) - \sum_{\substack{\varrho\in Z(\zeta)\\ |\mathfrak{Im}\varrho|<T}}\Phi(-\varrho)$, instead of (6.5), and we use the functional equation $-\frac{\zeta'}{\zeta}(s) = -\log\pi + \frac{\zeta'}{\zeta}(1-s) + \frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right)\right)$ of [5, page 59] instead of (6.6). $\square$

6.2. **Proof of Proposition 3.2.** Let $s \in \mathbb{C}$. We consider $\phi(y) = (f(0) - f(y))e^{-ys}$ if $y \geq 0$, and $\phi(y) = 0$ otherwise. Thus ϕ verifies all the conditions stated in Theorem 6.1. We have $\phi(0) = 0$, $\phi(-\log n) = 0$, and for each $\mathfrak{Re} z < \mathfrak{Re} s$, $\Phi(-z) = \frac{f(0)}{s-z} - F(s-z) = -\frac{F_2(s-z)}{(s-z)^2}$. We insert this definition in the explicit formula (6.1), and take its real part. Together with the classical explicit formula [5, Equations (17) (18) on p 83] for $-\mathfrak{Re}\frac{L'}{L}(s,\chi)$, namely

$$\mathfrak{Re}\sum_{n\geq 1}\frac{\Lambda(n)\chi(n)}{n^s} = \frac{1}{2}\log\frac{q}{\pi} + \frac{1}{2}\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) - \sum_{\varrho\in Z(\chi)}\mathfrak{Re}\frac{1}{s-\varrho}, \tag{6.12}$$

we obtain $$f(0)\Big(\frac{1}{2}\log\frac{q}{\pi} + \frac{1}{2}\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) - \sum_{\varrho\in Z(\chi)}\mathfrak{Re}\frac{1}{s-\varrho}\Big) - \mathfrak{Re}\sum_{n\geq 1}\frac{\Lambda(n)\chi(n)}{n^s}f(\log n)$$

$$= -f(0)\sum_{\varrho\in Z(\chi)}\mathfrak{Re}\frac{1}{s-\varrho} + \sum_{\varrho\in Z(\chi)}\mathfrak{Re}F(s-\varrho) - \mathfrak{Re}\frac{1}{2i\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{z+\mathfrak{a}}{2}\right)\frac{F_2(s-z)}{(s-z)^2}dz.$$

This establishes (3.21) for $\mathfrak{Re} s > 1$. Since the functions defined on the left and right hand side of the equality are both harmonic functions defined on the whole complex plane, then the identity extends to all $s \in \mathbb{C}$.

We prove (3.22) in a similar manner: we use (6.2) with $\Phi(-1) = \frac{f(0)}{s-1} - F(s-1), \Phi(0) = -\frac{F_2(s)}{s^2}$, and the classical explicit formula [5, (8) (11) page 80]

$$\mathfrak{Re}\sum_{n\geq 1}\frac{\Lambda(n)}{n^s} = \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\mathfrak{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s+2}{2}\right) - \sum_{\varrho\in Z(\zeta)}\mathfrak{Re}\frac{1}{s-\varrho}. \tag{6.13}$$

6.3. **Handling the non-primitive characters.** We introduce similar notation to [24, Section 3.4.]:

$$c_p(\sigma,\kappa,\delta) = \sum_{m\geq 1}\frac{1}{p^{m\sigma}}\left(1 - \frac{\kappa}{p^{m\delta}}\right) = \frac{1}{p^\sigma - 1} - \frac{\kappa}{p^{\sigma+\delta}-1}. \tag{6.14}$$

Lemma 6.2. *Let $\sigma, \gamma_0, \kappa, \delta, t_0$ be as in Section 3.1 and Proposition 5.2. Let $n_0 \geq 2$ and $P(x) = \sum_{k=0}^{n_0} a_k$ be the trigonometric polynomial as in Section 5.6. In addition we assume that these coefficients and σ_0, κ, δ satisfy* (3.12)

$$a_0 + \left(\frac{1-\kappa}{\max\left(c_2(\sigma_0,\kappa,\delta), 2c_3(\sigma_0,\kappa,\delta)\right)} - 1\right)\sum_{k=2}^{n_0} a_k > 0. \tag{6.15}$$

Then

$$\frac{1-\kappa}{2}f(0)\sum_{k=1}^{n_0} a_k\log\left(\frac{q}{q_k}\right) + \sum_{k=0}^{n_0} a_k S(\sigma + ik\gamma_0, \chi_{(k)} - \chi^k) \geq 0. \tag{6.16}$$

Proof. We input the definition of $S(s,\chi)$ in (6.16). Note that for $k=1$ the factor of a_1 vanishes. The left term equals

$$a_0\sum_{p|q}\sum_{m\geq 1}\frac{\log p}{p^{m\sigma}}f(m\log p)\left(1 - \frac{\kappa}{p^{m\delta}}\right) + \sum_{k=2}^{n_0} a_k\Bigg(\frac{1-\kappa}{2}f(0)\log\frac{q}{q_k}$$

$$+ \sum_{\substack{p|q\\ p\nmid q_k}}\sum_{m\geq 1}\mathfrak{Re}\left(\frac{\chi_{(k)}(p^m)}{p^{imk\gamma_0}}\right)\frac{(\log p)f(m\log p)}{p^{m\sigma}}\left(1 - \frac{\kappa}{p^{m\delta}}\right)\Bigg). \tag{6.17}$$

For $k \geq 2$, we use the inequalities $\log \frac{q}{q_k} \geq \sum_{\substack{p|q \\ p\nmid q_k}} \nu_p\left(\frac{q}{q_k}\right) \log p$, $f(0) \geq \sum_{m\geq 1} \frac{f(m\log p)}{p^{m\sigma} c_p(\sigma,\kappa,\delta)}\left(1-\frac{\kappa}{p^{m\delta}}\right)$, and $\mathfrak{Re}\left(\frac{\chi_{(k)}(p^m)}{p^{imk\gamma_0}}\right) \geq -1$, where ν_p is the notation for p-adic valuation. Using the notation

$$C_p(\sigma,\kappa,\delta) = a_0 + \sum_{k=2}^{n_0} a_k \left(\frac{1-\kappa}{2c_p(\sigma,\kappa,\delta)}\nu_p\left(\frac{q}{q_k}\right) - 1\right) > 0,$$

we have that (6.17) is larger than

$$\sum_{p|q}\sum_{m\geq 1} \frac{(\log p) f(m\log p)}{p^{m\sigma}}\left(1-\frac{\kappa}{p^{m\delta}}\right) C_p(\sigma,\kappa,\delta).$$

It is immediate that $c_p(\sigma)$ is positive and increases as δ increases and decreases as κ increases. It also it decreases as p or σ increases: this is easily verified by showing that the respective derivatives have opposite sign of $\sigma\left(1-p^{-(\sigma+\delta)}\right)^2 - \kappa(\sigma+\delta)p^{-\delta}\left(1-p^{-\sigma}\right)^2$, which is positive under the conditions (3.10). It follows that

$$c_p(\sigma,\kappa,\delta) \leq \begin{cases} 1.014351 & \text{if } p=2, \\ 0.533948 & \text{if } p\geq 3, \end{cases}$$

Note that since there is no primitive characters modulo 2, there is no modulus q with primitive characters of the form $2q'$ with q' odd. Thus $\nu_2\left(\frac{q}{q_k}\right) \geq 2$. Since $\sigma \geq 0.9, 0.5 \leq \delta \leq 0.75, 0.25 \leq \kappa \leq 0.5$, then

$$\frac{1-\kappa}{2c_p(\sigma,\kappa,\delta)}\nu_p\left(\frac{q}{q_k}\right) - 1 \geq \begin{cases} -0.507074 & \text{if } p=2, \\ -0.531790 & \text{if } p\geq 3. \end{cases}$$

and

$$C_p(\sigma,\kappa,\delta) \geq a_0 + \begin{cases} \min\left(0, \frac{1-\kappa}{c_2(\sigma_0,\kappa,\delta)} - 1\right)\sum_{k=2}^{n_0} a_k & \text{if } p=2, \\ \min\left(0, \frac{1-\kappa}{2c_3(\sigma_0,\kappa,\delta)} - 1\right)\sum_{k=2}^{n_0} a_k & \text{if } p\geq 3. \end{cases}$$

We assumed that these quantities were positive, which achieves the proof. □

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