AN EXPLICIT ZERO-FREE REGION FOR DIRICHLET L-FUNCTIONS

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Abstract: Let $\mathcal{L}_q(s)$ be the product of Dirichlet L-functions modulo q. Then $\mathcal{L}_q(s)$ has at most one zero in the region
$$\Re s \geq 1 - \frac{1}{6.41 \log \max(q, q|\Im s|)}.$$

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1. Introduction

Let q be a positive integer, χ a non-principal primitive character modulo q, and $L(s,\chi)$ the associated Dirichlet L-function. We recall that the Dirichlet series representation

$$L(s,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n^s}$$

defines a function holomorphic for $\Re s>0$. It can be holomorphically continued to the whole complex plane and it never vanishes in $\Re s>1$. Its zeros in the half plane $\Re s\leq 0$ are the integers $-\mathfrak{a}-2n$, $n\geq 0$, where $\mathfrak{a}=(1-\chi(-1))/2$. All remaining zeros are in the the critical strip $0<\Re s<1$ and are distributed symmetrically with respect to the critical line $\Re s=1/2$. Also, if ρ is a zero of $L(s,\chi)$, then $\overline{\rho}$ is a zero of $L(s,\overline{\chi})$.

Let $\mathcal{L}_q(s)$ be the product of the $\phi(q)$ Dirichlet L-functions modulo q. In the case $q=1$, we have $\mathcal{L}_1(s)=\zeta(s)\prod_{p|q}(1-p^{-s})$, where $\zeta(s)$ is the Riemann zeta function. In his proof of the Prime Number Theorem, de la Vallée Poussin established that there exists a constant R such that $\zeta(\sigma+it)\neq 0$ when

$$\sigma\geq 1-\frac{1}{R\log|t|}\qquad(|t|\geq 2)\ . \tag{1.1}$$

Explicit estimates of the constants R have been given by de la Vallée Poussin, Landau, Stechkin, and Rosser and Schoenfeld who last found $R=9.65$ in 1975 (see [10]). The author showed in [6] that the value $R=5.70$ is valid. It turns out that the same techniques would extend to (1.1) the case $q>1$. It follows from papers by Gronwall and by Titchmarsh that $\mathcal{L}_q(\sigma+it)$ has at most one zero in the region

$$\sigma\geq 1-\frac{1}{R_0\log q(|t|+2)}\ .$$

There have been several investigations of R_0, the latest given by McCurley in 1984 (see [8]) with $R_0=9.65$. We prove the following theorems:

Theorem 1.1. *The function $\mathcal{L}_q(\sigma+it)$ has, at most, one single zero in the region:*

$$\sigma\geq 1-\frac{1}{R_0\log\left(\max(q,q|t|)\right)},\quad \text{where } R_0=6.41.$$

Such a zero, if it exists, is real, simple and corresponds to a real non-principal character modulo q. We shall refer to it as an exceptional zero and q as an exceptional modulus.

We shall describe more precisely the case of an exceptional zero with the two following theorems. The first describes explicitly the phenomenon of repulsion that exceptional zeros exhibit.

Theorem 1.2. *If χ is a real primitive character modulo q and if β_1 and β_2 are two real zeros of $L(s,\chi)$, then:*

$$\min(\beta_1,\beta_2)\leq 1-\frac{1}{R_1\log q},\ \text{where } R_1=1.10.$$

Theorem 1.3. *If χ_1 and χ_2 are two distinct real primitive characters modulo q_1 and q_2 respectively and if β_1 and β_2 are real zeros of $L(s,\chi_1)$ and $L(s,\chi_2)$ respectively, then:*

$$\min(\beta_1,\beta_2)\leq 1-\frac{1}{R_2\log(q_1q_2)},\ \text{where } R_2=2.05.$$

These results improve on McCurley's constants from [7]: $R_1=1.62$ and $R_2=3.23$. When $q_1=q_2$ is an exceptional modulus, Theorem 1.3 shows that the exceptional zero repels the other real zeros of the exceptional conductor. More precisely, the region $\sigma\geq 1-\frac{1}{4.10\log q}$ and $t=0$ contains at most one zero. Now, assuming $q_1<q_2$, then the inequality implies that both q_1 and q_2 cannot be exceptional, unless $q_2\geq q_1^{2.12}$.
We remark that Heath-Brown, in his research concerning Linnik's constant (in [5]), established $R_0=2.88$ for q asymptotically large and $|t|\leq 1$. Also Wang Wei reduces it

to $R_0 = 2.35$, in [13], in the case where $|t| \le \log\log\log q$ and q asymptotically large. One of the key points of Heath-Brown's proof is an improvement of Burgess' bounds for character sums. This is where the condition on q being asymptotically large is imposed. Here, we shall employ another strategy since we are aiming to obtain a result valid for all q.

We now outline the principal ideas of the proof of the theorems. Let $\varrho_0 = \beta_0 + i\gamma_0$ denote the zero we want to locate. First, in section 2.1, we establish a version of Weil's formula relating the zeros of $L(s,\chi)$ to prime numbers. Let f be a positive smooth function chosen such that its Laplace transform $F(z) = \int_0^{+\infty} e^{-zt} f(t)\,dt$ satisfies:

$$F(z) = \frac{f(0)}{z} + \mathcal{O}\left(\frac{1}{|z|^2}\right) \quad \text{and} \quad \Re F(z) \ge 0 \text{ for } \Re z \ge 0. \tag{1.2}$$

Then

$$\Re \sum_{n\ge 1} \frac{\Lambda(n)\chi(n)}{n^s} f(\log n) = \frac{f(0)}{2}\log\max(|t|,1) + \frac{\delta_\chi}{2}\log q + \delta_\chi \Re F(s-1) - \sum_{\varrho\in Z(\chi)} \Re F(s-\varrho) + R(s), \tag{1.3}$$

where $\delta_\chi = 1$ if χ is principal and $\delta_\chi = 0$ otherwise, $Z(\chi)$ is the set of non-trivial zeros of $L(s,\chi)$, and $R(s)$ is an error term. Note that for $f = 1$ and χ non-principal, (1.3) reduces to

$$-\Re\frac{L'}{L}(s,\chi) = \frac{1}{2}\log\left(q\max(|t|,1)\right) + \delta_\chi \Re\frac{1}{s-1} - \sum_{\varrho\in Z(\chi)} \Re\frac{1}{s-\varrho} + R(s).$$

In section 2.3 we shall make a specific choice of f. In (1.3) the $\log q$-term arises from the size of the logarithmic derivative of $L(s,\chi)$. One of the key points in reducing the value of R_0 is to reduce the coefficient $1/2$ arising. For example, the new Burgess bound of Heath-Brown leads to a value of $1/4$. In our argument we will approach a limiting value $(1-\kappa)/2 \simeq 0.29$. The intuitive idea, which stems from an argument of de la Vallée Poussin, is to compare the size of our L-function at different points $s_k = \sigma + ik\gamma_0$, $k = 0,1,2,...,d$, on a line near $\Re s = 1$:

- when $k=0$, s_0 is close to the pole of zeta,
- when $k=1$, s_1 is close to the zero $\beta_0 + i\gamma_0$,
- when $k \ge 2$, the L-function is bounded at s_k.

The comparison is rendered possible by using trigonometrical inequalities of the type:

$$\sum_{k=0}^{d} a_k \cos(k\gamma_0) \ge 0 \quad \text{and all} \quad a_k \ge 0.$$

In section 2.2, we present our choice of polynomial. Combined with the left hand term of (1.3), we obtain:

$$\sum_{n\ge 1} \frac{\Lambda(n)}{n^\sigma} f(\log n) \sum_{k=0}^{d} a_k \cos\left(\arg\left(\frac{\chi^k(n)}{n^{ik\gamma_0}}\right)\right) \ge 0.$$

This gives different right hand terms for (1.3). When $k = 0$, 1 or is ≥ 2, we obtain

$$\Re F(\sigma-1) + R_1(s), \tag{1.4}$$

$$\frac{f(0)}{2}\log\left(q\max(|\gamma_0|,1)\right) - \Re F(s-\varrho_0) + R_2(s), \tag{1.5}$$

$$\frac{f(0)}{2}\log\left(q\max(|\gamma_0|,1)\right) + R_3(s), \tag{1.6}$$

respectively. The $R_j(s)$ are just small error terms. In each case, we use one major argument which concerns the size of the sum over the zeros. We show that, thanks to our choice of F as in (1.2), we can control its size so that it is not bigger than the error term. The details are provided in sections 7 to 10. We also add another ingredient to the proof, which is to consider not only the points listed as above but also the ones a little bit further to the left. Instead of

$$\Re \sum_{n\geq 1} \frac{\Lambda(n)\chi(n)}{n^s} f(\log n) \ ,$$

we study

$$\Re \sum_{n\geq 1} \frac{\Lambda(n)\chi(n)}{n^s} f(\log n) \left(1 - \frac{\kappa}{n^\delta}\right)$$

and its related trigonometric inequality. The positive constants δ and κ will be chosen so that the new Laplace transform term $F(s) - \kappa F(s+\delta)$ satisfies (1.2). Therefore the size of the sum over the zeros remains small. In section 7 we will detail this argument inspired by a lemma due to Stechkin and used by Rosser and Schoenfeld and then McCurley to improve the classical Riemann's zero-free region. Note that this is where the reduction from $1/2$ to $(1-\kappa)/2$ arises. Putting together all these arguments leads to the inequality:

$$0 \leq a_0 \Re F(\sigma - 1) - a_1 \Re F(\sigma - \beta_0) + \frac{1-\kappa}{2}(a_1 + ... + a_d) f(0) \log\left(q \max(|\gamma_0|, 1)\right). \tag{1.7}$$

We choose f to depend on β_0 by setting $f(0) = c_0(1-\beta_0)$, with c_0 constant, and we deduce from (1.7) that

$$(1-\beta_0)\ \log\left(q \max(|\gamma_0|, 1)\right) \geq \frac{2}{1-\kappa}\ \frac{a_1 \Re F(\sigma - \beta_0) - a_0 \Re F(\sigma - 1)}{c_0(a_1 + ... + a_d)}.$$

We conclude by optimizing the right term with respect to σ and we obtain a computable value for $\frac{1}{R_0}$.

2. Preliminaries

2.1. **An explicit formula.** Let f be a function that satisfies the following properties : f is a positive function in $\mathcal{C}^2([0,d])$, with compact support in $[0,d]$ and

$$f(d) = f'(0) = f'(d) = f''(d) = 0. \tag{2.1}$$

We denote F its Laplace transform

$$F(s) = \int_0^d e^{-st} f(t) dt$$

and F_2 the Laplace transform of f''.
We will define f explicitly in section 2.3. In Theorem 3.1 of [6], we gave an explicit formula relating sums over zeros of a Dirichlet L-function and sums over primes. It is a special version of the Guinand-Riemann-Weil formula established for Hecke L-functions (see [14]).

Proposition 2.1. *Let f be a function satisfying (2.1) and $s = \sigma + it$ a complex number. We denote $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$. We have:*

$$\begin{aligned}\Re \sum_{n\geq 1} \frac{\Lambda(n)}{n^s} f(\log n) = f(0)\left(-\frac{\log \pi}{2} + \frac{1}{2}\Re\psi\left(\frac{s}{2}+1\right)\right) \\ + \Re F(s-1) - \sum_{\varrho \in Z(\zeta)} \Re F(s-\varrho) + \Re \frac{F_2(s)}{s^2} \\ + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Re\psi\left(\frac{1}{4} + i\frac{T}{2}\right) \Re \frac{F_2(s-1/2-iT)}{(s-1/2-iT)^2} dT,\end{aligned} \tag{2.2}$$

where $Z(\zeta)$ denotes the set of non-trivial zeros of $\zeta(s)$. If χ is a primitive non-principal character modulo q, then:

$$\Re\sum_{n\geq 1}\frac{\Lambda(n)\chi(n)}{n^s}f(\log n) = f(0)\left(\frac{\log(q/\pi)}{2}+\frac{1}{2}\Re\psi\left(\frac{s+\mathfrak{a}}{2}\right)\right) - \sum_{\varrho\in Z(\chi)}\Re F(s-\varrho)+\frac{1-\mathfrak{a}}{2}\Re\frac{F_2(s)}{s^2} + \frac{1}{2\pi}\int_{\mathbb{R}}\Re\psi\left(\frac{2\mathfrak{a}+1}{4}+i\frac{T}{2}\right)\Re\frac{F_2(s-1/2-iT)}{(s-1/2-iT)^2}dT, \tag{2.3}$$

where $Z(\chi)$ denotes the set of non-trivial zeros of $L(s,\chi)$

Proof. Both identities are deduced from Theorem 3.1 of [6]. In particular, formula (2.2) is coming from Proposition 2.1 of [6]. Formula (2.3) is obtained the same way, together with the classical explicit formula (see Chapter 14 of [3]):

$$-\Re\frac{L'}{L}(s,\chi) = \frac{1}{2}\log\frac{q}{\pi}+\frac{1}{2}\Re\psi\Big(\frac{s+\mathfrak{a}}{2}\Big) - \sum_{\varrho\in Z(\chi)}\Re\frac{1}{s-\varrho}.$$

□

Let κ and δ be real numbers in $[0,1]$. We introduce

$$S(s)=\Re\sum_{n\geq 1}\frac{\Lambda(n)}{n^s}f(\log n)\left(1-\frac{\kappa}{n^\delta}\right),\ S(s,\chi)=\Re\sum_{n\geq 1}\frac{\Lambda(n)\chi(n)}{n^s}f(\log n)\left(1-\frac{\kappa}{n^\delta}\right).$$

2.2. **A trigonometric inequality.** Let P be a trigonometric polynomial of degree d:

$$P(x)=\sum_{k=0}^{d}a_k\cos(kx)\geq 0, \tag{2.4}$$

with every $a_k>0$. De la Vallée Poussin used degree 2 with:

$$P(x)=2(1+\cos x)^2=3+4\cos x+\cos(2x)$$

and Rosser and Schoenfeld extended the degree to 4 in [10]:

$$P(x)=8(0.9126+\cos x)^2(0.2766+\cos x)^2.$$

Both were optimal choices for each degree. In the author's previous article [6] and here, we consider a slighty modified form of the preceeding polynomial:

$$P(x)=\sum_{k=0}^{4}a_k\cos(kx)=8(0.91+\cos x)^2(0.265+\cos x)^2\geq 0, \tag{2.5}$$

where

$$a_0=10.91692658,\ a_1=18.63362,\ a_2=11.4517,\ a_3=4.7,\ a_4=1, \tag{2.6}$$

and $A=\sum_{k=1}^4 a_k=35.78532$. This seems to be not far from optimal for our choice of the smooth weight f.
We apply (2.4) to compare the values of $S(\sigma+ik\gamma_0,\chi^k)$:

$$\sum_{k=0}^{4}a_kS(\sigma+ik\gamma_0,\chi^k)=\sum_{n\geq 1}\frac{\Lambda(n)f(\log n)}{n^\sigma}\Big(1-\frac{\kappa}{n^\delta}\Big)P\Big(\arg\left(\frac{\chi(n)}{n^{i\gamma_0}}\right)\Big)\geq 0. \tag{2.7}$$

Let $\chi_{(k)}$ be the primitive character associated with χ^k of conductor q_k.

$$\sum_{k=0}^{4}a_kS(\sigma+ik\gamma_0,\chi^k)=\sum_{k=0}^{4}a_kS(\sigma+ik\gamma_0,\chi_{(k)})-\sum_{k=0}^{4}a_kS(\sigma+ik\gamma_0,\chi_{(k)}-\chi^k).$$

All characters $\chi_{(k)}$ are primitive and therefore we can apply Proposition 2.1 to them. We now treat the last term.

In lemma 10.1, we will establish that for our choice of the function f, of the coefficients a_i's and of κ, δ and σ, we have:

$$\frac{1-\kappa}{2} f(0) \sum_{k=1}^{4} a_k \log\left(\frac{q}{q_k}\right) + \sum_{k=0}^{4} a_k S(\sigma + ik\gamma_0, \chi_{(k)} - \chi^k) \geq 0. \tag{2.8}$$

By adding (2.7) and (2.8), we conclude that:

$$\Sigma_1 = \sum_{k=0}^{4} a_k S(\sigma + ik\gamma_0, \chi_{(k)}) + \frac{1-\kappa}{2} f(0) \sum_{k=1}^{4} a_k \log\left(\frac{q}{q_k}\right) \geq 0. \tag{2.9}$$

We rewrite this in the case when the order of χ is at most four:

- If χ is of order 4:

$$\begin{aligned} \Sigma_1 = a_0 S(\sigma) + a_1 S(\sigma + i\gamma_0, \chi) + a_2 S(\sigma + 2i\gamma_0, \chi_{(2)}) \\ + a_2 \frac{1-\kappa}{2} f(0) \log \frac{q}{q_2} + a_3 S(\sigma + 3i\gamma_0, \overline{\chi}) + a_4 S(\sigma + 4i\gamma_0) \geq 0. \end{aligned} \tag{2.10}$$

- If χ is of order 3:

$$\begin{aligned} \Sigma_1 = a_0 S(\sigma) + a_1 S(\sigma + i\gamma_0, \chi) + a_2 S(\sigma + 2i\gamma_0, \chi_{(2)}) \\ + a_3 S(\sigma + 3i\gamma_0) + a_4 S(\sigma + 4i\gamma_0, \chi) \geq 0. \end{aligned} \tag{2.11}$$

- If χ is of order 2:

$$\begin{aligned} \Sigma_1 = a_0 S(\sigma) + a_1 S(\sigma + i\gamma_0, \chi) + a_2 S(\sigma + 2i\gamma_0) \\ + a_3 S(\sigma + 3i\gamma_0, \chi) + a_4 S(\sigma + 4i\gamma_0) \geq 0. \end{aligned} \tag{2.12}$$

Note that, when the zero ϱ_0 is close to the real axis, though not real, it is also close to $\overline{\varrho_0}$, which is a zero associated to the conjugate character. It will also be useful to have inequalities involving both $S(\sigma, \chi)$ and $S(\sigma, \overline{\chi})$. If χ^j is principal, then

$$\Re\left(1 + \chi(n) + \chi^2(n) + ... + \chi^{j-1}(n)\right) \geq 0. \tag{2.13}$$

We apply this inequality and (2.8) in the cases where

- χ is of order 4. Here $a_0 = 1,\ a_1 = 2,\ a_2 = 1,\ a_3 = a_4 = 0$ and

$$\Sigma_2 = \frac{1-\kappa}{2} f(0) \log \frac{q}{q_2} + S(\sigma) + S(\sigma, \chi) + S(\sigma, \overline{\chi}) + S(\sigma, \chi_{(2)}) \geq 0. \tag{2.14}$$

- χ is of order 3. Here $a_0 = 1,\ a_1 = 2,\ a_2 = a_3 = a_4 = 0$ and

$$\Sigma_3 = S(\sigma) + S(\sigma, \chi) + S(\sigma, \overline{\chi}) \geq 0. \tag{2.15}$$

- χ is real. Here $a_0 = 1,\ a_1 = 2,\ a_2 = a_3 = a_4 = 0$ and

$$\Sigma_4 = S(\sigma) + S(\sigma, \chi) \geq 0. \tag{2.16}$$

Let χ_1 and χ_2 be real primitive characters. Since the product $(1 + \chi_1(n))(1 + \chi_2(n))$ is positive, then:

$$\sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma} \left(1 - \frac{\kappa}{n^\delta}\right) (1 + \chi_1(n))(1 + \chi_2(n)) \geq 0.$$

We denote the above sum as Σ_5. Then:

$$\Sigma_5 = S(\sigma) + S(\sigma, \chi_1) + S(\sigma, \chi_2) + S(\sigma, \chi_1\chi_2) \geq 0. \tag{2.17}$$

In the next two sections we introduce a specific function f and several other parameters.

2.3. **The test function.**

2.3.1. *Definition.* We call F the Laplace transform of f and $\tilde{F}(x,y)$ the real part of $F(x+iy)$:

$$\tilde{F}(x,y) = \int_0^d f(t)e^{-xt}\cos(yt)d\,t.$$

Now we give an explicit definition of f such that it satisfies (2.1) and such that F satisfies:

$$\tilde{F}(x,y) \geq 0 \quad \text{if} \quad x \geq 0. \tag{2.18}$$

Let $\theta \in]\pi/2, \pi[$ be fixed and let g_θ be the positive function:

$$g_\theta(x) = \begin{cases} (1+\tan^2\theta)\left(\cos\left(x\tan\theta\right) - \cos\theta\right) & \text{if } \dfrac{\theta}{\tan\theta} \leq x \leq \dfrac{-\theta}{\tan\theta}, \\ 0 & \text{otherwise.} \end{cases}$$

By a calculus of variations, Heath-Brown chose $h_\theta = g_\theta * g_\theta$ and suggested (see lemma 7.4 in [5]) that the function

$$f(t) = f_{\eta,\theta}(t) = \eta h_\theta(\eta t)$$

may not be far from optimal under our conditions. The function h_θ is explicitly

$$h_\theta(u) = (1+\tan^2\theta)\Big[(1+\tan^2\theta)\Big(\frac{-\theta}{\tan\theta} - \frac{u}{2}\Big)\cos(u\tan\theta) - \frac{2\theta}{\tan\theta} \\ - u - \frac{\sin(2\theta + u\tan\theta)}{\sin(2\theta)} + 2\Big(1 + \frac{\sin(\theta + u\tan\theta)}{\sin\theta}\Big)\Big],$$

when $u \in [0, d_1(\theta)]$, $d_1(\theta) = \frac{-2\theta}{\tan\theta}$, and $h_\theta(u) = 0$ otherwise. We have the special value

$$f(0) = \eta h_\theta(0) \quad \text{with} \quad h_\theta(0) = (1+\tan^2\theta)(3 - \theta\tan\theta - 3\theta\cot\theta).$$

Note that $h_\theta(u)$ decreases with u since the derivative

$$\int_{-\infty}^{+\infty} g_\theta'(u-x)g_\theta(x)d\,x \\ = -\tan\theta(1+\tan^2\theta)^2\int_{u+\frac{\theta}{\tan\theta}}^{-\frac{\theta}{\tan\theta}} \sin\left((u-x)\tan\theta\right)\left(\cos\left(x\tan\theta\right) - \cos\theta\right)d\,x$$

is negative. It is a standard calculation to verify that the integral is negative by considering the cases $-\frac{\theta}{\tan\theta} \leq u \leq -\frac{2\theta}{\tan\theta}$ and $0 \leq u \leq -\frac{\theta}{\tan\theta}$.

2.3.2. *Properties of the Laplace transform.* In this section, we recall the approximations for $\tilde{F}$ that we proved in [6]. Let F_2 be the Laplace transform of f''. Since f satisfies (2.1), we have

$$F(s) = \frac{f(0)}{s} + \frac{F_2(s)}{s^2}. \tag{2.19}$$

We take the real part of this and observe that

$$\tilde{F}(x,y) = \frac{x}{x^2+y^2}\,h_\theta(0)\eta + H(x,y) \text{ where } H(x,y) = \Re\left(\frac{F_2(x+iy)}{(x+iy)^2}\right). \tag{2.20}$$

We recall two bounds on H, established in [6]:

Lemma 2.2. *Let*

$$M(z) = \int_0^{d_1(\theta)} |h_\theta''(u)|\,e^{-zu}du \text{ and } m_\theta = \max_{u\in[0,d_1(\theta)]} |h_\theta''(u)| = |h_\theta''(0)|.$$

For any real numbers x and y, we have:

$$|H(x,y)| \leq \frac{M\left(x/\eta\right)}{x^2+y^2}\,\eta^2, \tag{2.21}$$

If $x \geq 0$, then:

$$|H(x,y)| \leq \frac{m_\theta}{x(x^2+y^2)}\,\eta^3. \tag{2.22}$$

2.4. **Notation.** Let q be an integer larger than $q_0 \geq 3$ and let χ be a non-principal primitive character of conductor q. The aim of this article is to locate the zeros of $L(s,\chi)$. We fix one zero, $\varrho_0 = \beta_0 + i\gamma_0$. We can choose $\gamma_0 \geq 0$, since the zeros of $L(s,\chi)$ are symmetric about the real line with those of $L(s,\overline{\chi})$. We assume that ϱ_0 satisfies

$$1 - \frac{1}{5\log\left(q\max\left(1,|\gamma_0|\right)\right)} \leq \beta_0 < 1 - \frac{1}{R\log\left(q\max\left(1,|\gamma_0|\right)\right)}$$

where R is a positive constant for which the zero-free region :

$$\Re s \geq 1 - \frac{1}{R\log\left(q\max(1,|\Im s|)\right)}$$

is true. The result of McCurley in [7] allows us to commence with $R = 9.645908801$.

2.4.1. *Introducing the variables η, σ, t_0 and ω.* We set

$$\eta = 1 - \beta_0 = \frac{1}{r\log\left(q\max\left(1,\gamma_0\right)\right)} \text{ where } 5 \leq r \leq R$$

and

$$\sigma = 1 - \frac{1}{R\log\left(q(4\max(1,\gamma_0)+t_0)\right)},$$

where $t_0 \geq 1$ is a parameter that shall be chosen later. We chose $\sigma > \beta_0$, so that all the points $\sigma + ik\gamma_0$ are on the right of the points $\beta_0 + ik\gamma_0$. We have the bounds $0 < \eta \leq \eta_0$ and $\sigma \geq \sigma_0$, where

$$\eta_0 = \begin{cases} \frac{1}{r\log q_0}, \\ \frac{1}{r\log(q_0T_0)}, \end{cases} \qquad 1-\sigma_0 = \begin{cases} \frac{1}{R\log(q_0(4+t_0))} & \text{for any } \gamma_0 \geq 0, \\ \frac{1}{R\log(q_0(4T_0+t_0))} & \text{if } \gamma_0 \geq T_0 \geq 1. \end{cases} \tag{2.23}$$

Moreover, we put

$$\omega = \frac{1-\sigma}{\eta} = \frac{r\log\left(q\max(1,\gamma_0)\right)}{R\log\left(q(4\max(1,\gamma_0)+t_0)\right)} \text{ and } \omega_0 = \frac{1-\sigma_0}{\eta_0}.$$

The inequality $\omega \geq \omega_0$ comes from the fact that ω is a function increasing with each variable $m = \max(1,|\gamma_0|) \geq m_0 = \max(1,T_0)$ and $q \geq q_0$. Note that ω_0 lies in $[0.01, 1]$.

2.4.2. *Introducing the variables κ and δ.* The motivation for introducing κ and δ will be given in section 7.2. The Proposition 7.4 also provides definitions for them. They depend on θ, r, R and q (however, the numerical values are rather stable). We now give some of the properties they will have to satisfy:

$$0.25 < \kappa < 0.45 \text{ and } 0 < \delta < 1, \tag{2.24}$$

$$\left(\delta^{-3} + \left(1-\eta_0+\delta\right)^{-3}\right)^{-1} \leq \kappa \leq \left(\delta^{-1} + \left(1-\eta_0+\delta\right)^{-1}\right)^{-1}. \tag{2.25}$$

2.4.3. *Condition for the choice of θ.* In the next sections, the we will need to find a lower bound for the difference:

$$a_1\tilde{F}(\sigma-\beta_0,0) - a_0\tilde{F}(\sigma-1,0).$$

Note that it depends only on a_1, a_0, ω and θ since it equals

$$\int_0^{d_1(\theta)} \left(a_1e^{-t} - a_0\right) h_\theta(t) e^{\omega t} d\,t$$

We denote $K(\omega,\theta)$ this integral.

Lemma 2.3. *Let $\omega_1 = 0.01$ and $\omega_2 = 1$. The function $K(\omega,\theta)$ increases with ω in $[\omega_1,\omega_2]$ when*

$$\begin{cases} \theta \in]\pi/2, 1.899497[, a_0 = 10.91692658 \text{ and } a_1 = 18.63362, \\ \theta \in]\pi/2, 1.966320[, a_0 = 1 \text{ and } a_1 = 2. \end{cases}$$

Proof. $d(\theta)$ is increasing with θ. We denote θ_1 the point in $]\pi/2, \pi[$ such that, if $\theta \geq \theta_1$, then $a_1e^{-t} - a_0$ is positive for all $t \in [0, d_1(\theta_1)]$ and is negative for all $t \in [d_1(\theta_1), d_1(\theta)]$. We have:

$$\theta_1 = \begin{cases} 1.724582 & \text{when } a_0 = 10.91692658 \text{ and } a_1 = 18.63362, \\ 1.764719 & \text{when } a_0 = 1 \text{ and } a_1 = 2. \end{cases}$$

We deduce that the derivative $\frac{\partial K}{\partial \omega}(\omega, \theta)$ is larger than

$$\int_0^{d_1(\theta_1)} \left(a_1e^{-t} - a_0\right) h_\theta(t) t e^{\omega_1 t} dt + \int_{d_1(\theta_1)}^{d_1(\theta)} \left(a_1e^{-t} - a_0\right) h_\theta(t) t e^{\omega_2 t} dt. \tag{2.26}$$

By a GP-Pari calculation, we find the intervals for θ in which (2.26) is positive. □

The numerical values of $\theta, t_0, r, R, \kappa, \delta$ will be given in Tables 1 to 4.

2.4.4. *Notation for the terms appearing in the explicit formula.* We set:

$$\begin{aligned}
&D(s) = \Re F(s) - \kappa \, \Re F(s+\delta), \\
&D_1(\sigma+it) = -\frac{1-\kappa}{2}\log\pi + \frac{1}{2}\Re\psi\Big(\frac{\sigma+2+it}{2}\Big) - \frac{\kappa}{2}\Re\psi\Big(\frac{\sigma+2+\delta+it}{2}\Big), \\
&D_1(\sigma+it,\chi) = \frac{1-\kappa}{2}\log\frac{q}{\pi} + \frac{1}{2}\Re\psi\Big(\frac{\sigma+\mathfrak{a}+it}{2}\Big) - \frac{\kappa}{2}\,\Re\psi\Big(\frac{\sigma+\mathfrak{a}+\delta+it}{2}\Big), \\
&D_2(\sigma+it) = H(\sigma,t) - \kappa H(\sigma+\delta,t) \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}} \Re\psi\Big(\frac{1+2iT}{4}\Big)\Big(H\Big(\sigma-\frac{1}{2}, T-t\Big) - \kappa H\Big(\sigma-\frac{1}{2}+\delta, T-t\Big)dT, \\
&D_2(\sigma+it,\chi) = \frac{1-\mathfrak{a}}{2}\left(H(\sigma,t) - \kappa H(\sigma+\delta,t)\right) \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}} \Re\psi\Big(\frac{2\mathfrak{a}+1+2iT}{4}\Big)\Big(H(\sigma-\frac{1}{2}, T-t) - \kappa H(\sigma-\frac{1}{2}+\delta, T-t)\Big)dT.
\end{aligned}$$

We apply (2.2) and (2.3) to obtain respectively:

$$S(s) = f(0)D_1(s) + D(s-1) - \sum_{\varrho\in Z(\zeta)} D(s-\varrho) \; + D_2(s),$$

$$S(s,\chi) = f(0)D_1(s,\chi) - \sum_{\varrho\in Z(\chi)} D(s-\varrho) \; + D_2(s,\chi).$$

3. Case I: zeros of large imaginary part ($|\gamma_0| \geq 1$) or of character χ with large order (≥ 4).

The arguments that we are presenting in this section work for the three following situations:

- *Case I.A*: $L(s,\chi)$ satisfies GRH(T_0), that is to say $\gamma_0 \geq T_0$ where T_0 and q take the values listed in the table:

T_0	q	reference
10^4	$2, ..., 13,$	R. Rumely [9]
2 500	$14, ..., 72,$	
	$73, ..., 112$ and not prime,	
	$116, 117, 120, 121, 124, 125, 128,$	
	$132, 140, 143, 144, 156, 163, 169,$	
	$180, 216, 243, 256, 360, 420, 432.$	
10^4	$73, ..., 347$ and prime.	M. Bennett [1]

Note that for $2 \leq q \leq 113$ and $T_0 \geq 2500$, we have $qT_0 \geq 2 \cdot 10^4$.

- *Case I.B*: $q \geq 114$ and $\gamma_0 \geq 1$,
- *Case I.C*: $q \geq 114$, $\gamma_0 < 1$ and the order of χ is at least 4.

In each case, only the values of the implied parameters will change. The essential ingredient in the argument is the positivity of

$$\Sigma_1 = a_0 f(0) D_1(\sigma) + f(0) \sum_{k=1}^{4} a_k \Big(D_1(\sigma + ik\gamma_0, \chi_{(k)}) + \frac{1-\kappa}{2} \log \frac{q}{q_k} \Big) + a_0 D(\sigma - 1) - \sum_{k=0}^{4} a_k \sum_{\varrho \in Z(\chi_{(k)})} D(\sigma + ik\gamma_0 - \varrho) + \sum_{k=0}^{4} a_k D_2(\sigma, \chi_{(k)}). \tag{3.1}$$

We will study each of the terms on the right hand side in sections 7, 8 and 9. We now give an overview of the argument.

- The polar contribution of the Riemann zeta function at $s = 1$ gives rise to $a_0 D(\sigma - 1)$. It occurs that $\sigma - 1 + \delta$ is far from the pole. Therefore
$$a_0 D(\sigma - 1) = a_0 \big(\tilde{F}(\sigma - 1, 0) - \kappa \tilde{F}(\sigma - 1 + \delta, 0) \big) \simeq a_0 \tilde{F}(\sigma - 1, 0).$$
- The main contibution in the first line comes from the $D_1(\sigma + ik\gamma_0, \chi_{(k)})$ when $k \geq 1$. Since $\Re\psi(s) \simeq \log \max(|\Im s|, 1)$, we have that the fist line is equivalent to

$$f(0) \sum_{k=1}^{4} a_k \left(\frac{1-\kappa}{2} [\log q_k + \log \max(\gamma_0, 1)] + \frac{1-\kappa}{2} \log \frac{q}{q_k} \right) = f(0) A \frac{1-\kappa}{2} \log \left(q \max((\gamma_0, 1))\right)$$

- When $k = 1$ and $\varrho = \varrho_0$, $F(\sigma + ik\gamma_0 - \varrho) = \tilde{F}(\sigma - \beta_0, 0)$. We isolate this term from the sum over the zeros. Estimating the rest of the terms is delicate work explained in section 7. We obtain
$$\sum_{k=0}^{4} a_k \sum_{\varrho \in Z(\chi_{(k)})} D(\sigma + ik\gamma_0 - \varrho) \simeq a_1 \tilde{F}(\sigma - \beta_0, 0).$$
- The last term D_2 is negligible.

Lemma 3.1. *For the values of the parameters θ, t_0, r, R as listed in Table 1, we have the inequalities:*

$$0 \leq \Sigma_1 \leq \frac{A(1-\kappa) h_\theta(0)}{2} \eta \log \left(q \max(\gamma_0, 1)\right) + a_0 \tilde{F}(\sigma - 1, 0) - a_1 \tilde{F}(\sigma - \beta_0, 0).$$

We refer the reader to section 10 for the proof. We recall that $a_1 \tilde{F}(\sigma - \beta_0, 0) - a_0 \tilde{F}(\sigma - 1, 0) = K(\omega, \theta)$. We choose the value of θ as described in Lemma 2.3. Since $\omega \geq \omega_0$, we have

$$K(\omega, \theta) \geq K(\omega_0, \theta)$$

and

$$R_0 = R_0(R, r, t_0, \theta) = \frac{A(1-\kappa) h_\theta(0)}{2K(\omega_0, \theta)}, \tag{3.2}$$

is an acceptable constant for the zero-free region. Note that we have the dependences: $\kappa = \kappa(\eta_0, \sigma_0, \theta)$, $\omega_0 = \frac{1-\sigma_0}{\eta_0}$, $\eta_0 = \eta_0(r)$ and $\sigma_0 = \sigma_0(R, t_0)$. We refer the reader to section 10.2.2 for more details about the algorithm leading to the choice of θ, t_0, r, R and to the computation of R_0. We finally obtain:

Cases	*I.A*	*I.B*	*I.C*
R_0	5.847	6.250	6.246

4. *Case II*: ZEROS OF NOT TOO SMALL IMAGINARY PART ($\eta \ll |\gamma_0| < 1$) AND OF CHARACTER χ OF ORDER LESS THAN 4.

We call *Case II.A*, the situation when the order of χ is 4, *Case II.B* when it is 3 and *Case II.C* when χ is real. Since the order is smaller than the degree of the trigonometric polynomial, there exists some $k \geq 1$ such that χ^k is principal. Thus $S(\sigma + ik\gamma_0, \chi_k) = S(\sigma + ik\gamma_0)$ provides a new contribution which arises from the pole of zeta. In comparison with the *Case I.*, we lose the contribution that was given by $a_k D_1(\sigma + ki\gamma_0, \chi_k) \simeq a_k \frac{(1-\kappa)h_\theta(0)}{2} \log(q_k\gamma_0)\,\eta$. On the other hand, if we assume that γ_0 is bounded away from 0, namely if

$$\alpha\eta \leq |\gamma_0| < 1,$$

where α is a positive constant, $\alpha\eta = \mathcal{O}(1/\log q)$, then we gain a polar contribution from $a_k D(\sigma - 1 + ki\gamma_0)$ that balances our loss. Under this condition, *Case II* becomes analogous to *Case I* and we can establish:

Lemma 4.1. *Let α, θ, t_0, r, R take the values of Table 2. If $\alpha\eta \leq |\gamma_0| < 1$, then we have the inequalities:*

$$0 \leq \Sigma_1 \leq \frac{A(1-\kappa)h_\theta(0)}{2}\,\eta \log q + a_0 \tilde{F}(0,0) - a_1 \tilde{F}(1-\beta_0, 0).$$

We deduce the same definition (3.2) for R_0 and we obtain

$$\eta \log q \geq R_0^{-1}$$

with

Cases	*II.A*	*II.B*	*II.C*
α	2.6674	4.2748	7.9363
R_0	6.403	6.306	6.312

5. *Case III*: ZEROS OF SMALL IMAGINARY PART ($|\gamma_0| < \alpha\eta$) AND OF CHARACTER χ OF ORDER LESS THAN 4.

For technical convenience, we choose s on the real axis: $s = \sigma$. We now consider the case when γ_0 is closer to the real axis and consequently closer to the pole of ζ. Moreover, ϱ_0 is moving closer to its conjugate. To consider these two zeros, we use the trigonometric inequality for both χ and $\overline{\chi}$. To avoid the problem of poles arising from some possible

$$S(\sigma + ik\gamma_0, \chi_k) = S(\sigma + ik\gamma_0), \tag{5.1}$$

we change the trigonometric polynomial such that its degree does not exceed the order of the character, using the sums Σ_2, Σ_3, and Σ_4 for orders respectively equal to 4, 3, and 2. Doing so, the situation (5.1) cannot occur and the problem becomes closely analogous to the former cases. We end up with the sums

$$S(\sigma) \simeq D(\sigma - 1) \simeq \tilde{F}(\sigma - 1, 0),$$

$$S(\sigma, \chi) \simeq f(0)D_1(\sigma, \chi) - D(\sigma - \beta_0 - i\gamma_0) \simeq \frac{(1-\kappa)h_\theta(0)}{2}\,\eta \log q - \tilde{F}(\sigma - \beta_0, \alpha\eta),$$

$$S(\sigma, \chi_{(2)}) \simeq f(0)D_1(\sigma, \chi_{(2)}) \simeq \frac{(1-\kappa)h_\theta(0)}{2}\,\eta \log q_2.$$

Together with respectively (2.14), (2.15) and (2.16), we obtain:

Lemma 5.1. *Let α, θ, t_0, r, R be as listed in Table 3. If $0 \leq |\gamma_0| < \alpha\eta$, then*

- *Case III.A, when χ is of order 4:*

$$0 \leq \Sigma_2 \leq \frac{3(1-\kappa)h_\theta(0)}{2}\,\eta \log q + \tilde{F}(\sigma - 1, 0) - 2\tilde{F}(\sigma - \beta_0, \alpha\eta).$$

- *Case III.B, when χ is of order 3:*

$$0 \le \Sigma_3 \le (1-\kappa)h_\theta(0)\ \eta\log q + \tilde{F}(\sigma-1,0) - 2\tilde{F}(\sigma-\beta_0,\alpha\eta).$$

- *Case III.C, when χ is real:*

$$0 \le \Sigma_4 \le \frac{(1-\kappa)h_\theta(0)}{2}\ \eta\log q + \tilde{F}(\sigma-1,0) - 2\tilde{F}(\sigma-\beta_0,\alpha\eta).$$

We deduce that $\eta\log q \ge R_0^{-1}$ with

$$R_0 = \frac{A(1-\kappa)h_\theta(0)}{2}\left(\int_0^{d_1(\theta)} e^{\omega_0 t}h_\theta(t)\left(a_1e^{-t}\cos(\alpha t)-a_0\right)dt\right)^{-1}. \tag{5.2}$$

where $a_0 = 1$, $a_1 = 2$ and $A = 3, 2$ and 1 in cases *III.A*, *III.B* and *III.C* respectively.

Cases	*III.A*	*III.B*	*III.C*
α	2.6674	4.2748	7.9363
R_0	6.404	6.299	6.069

6. *Case IV.*: Case of exceptional characters.

It is widely expected that Dirichlet L-functions do not vanish on the real axis. Two recent results numerically affirming this are due to Watkins (see [12]) and Chua (see [2]) who respectively show that for all odd characters χ modulo $q \le 300\,000\,000$ and for all even characters χ modulo $q \le 200\,000$, the L-functions $L(s,\chi)$ do not vanish on the real axis. We can then choose $q \ge q_0 = 200\,000$. Let β_1 and β_2 be two real zeros such that $1/2 < \beta_2 \le \beta_1 \le 1$ and we set $1-\beta_2 = \eta$.

6.1. ***Case IV.A*: two real zeros associated to one real character.**

In this section we present the principal ideas of the proof of theorem 1.2. We use the trigonometric sum $\Sigma_4 = S(\sigma) + S(\sigma,\chi)$ and isolate the two zeros β_1 and β_2 of $L(s,\chi)$ in the sum $S(\sigma,\chi)$:

$$\tilde{F}(\sigma-\beta_1,0) + \tilde{F}(\sigma-\beta_2,0) \ge 2\tilde{F}(\sigma-\beta_2,0).$$

We establish together with (2.16):

Lemma 6.1. *For the values of the parameters θ, t_0, r, R as listed in Table 4, we have:*

$$0 \le \Sigma_4 \le \frac{(1-\kappa)h_\theta(0)}{2}\ \eta\log q + \tilde{F}(\sigma-1,0) - 2\tilde{F}(\sigma-\beta_2,0).$$

We deduce that $(1-\beta_2)\log q \ge R_1^{-1}$ where R_1's definition is given by (3.2) with $a_0 = 1$, $a_1 = 2$, $A = 1$. We obtain

$$R_1 = 1.098.$$

6.2. ***Case IV.B*: two real zeros associated to two real characters.**

In this section, we give the main lemma that induces Theorem 1.3. We isolate each zero β_1 and β_2 of $L(s,\chi_1)$ and $L(s,\chi_2)$ in the sums $S(\sigma,\chi_1)$ and $S(\sigma,\chi_2)$ appearing in

$$\Sigma_5 = S(\sigma) + S(\sigma,\chi_1) + S(\sigma,\chi_2) + S(\sigma,\chi_1\chi_2)$$

and together with (2.17), we obtain:

Lemma 6.2. *For the values of the parameters θ, r, t_0, κ and δ as listed in Table 4, we have:*

$$0 \le \Sigma_5 \le (1-\kappa)h_\theta(0)\ \eta\log(q_1q_2) + \tilde{F}(\sigma-1,0) - 2\tilde{F}(\sigma-\beta_2,0).$$

We deduce $(1-\beta_2)\log(q_1q_2) \ge R_2^{-1}$, where R_2's definition is given by (3.2) with $a_0 = 1$, $a_1 = 2$, $A = 2$. We find

$$R_2 = 2.042.$$

7. Study of the sum over the zeros.

In this section we fix the values of the parameters κ, δ and t_0. We need to study sums over zeros of the type:

$$\mathcal{S} = \sum_{\varrho \in Z(\chi')} D(\sigma + ik\gamma_0 - \varrho)$$

where χ' is a primitive character associated to q and where $0 \leq k \leq 4$. In particular, we will see in the next two following lemmas that in the case where χ' is the primitive character χ:

$$\mathcal{S} = \sum_{\varrho \in Z(\chi)} D(\sigma + ik\gamma_0 - \varrho) \simeq \Re F(\sigma - \beta_0), \text{ when } k = 1$$

$$\text{and } \mathcal{S} = \sum_{\varrho \in Z(\chi)} D(\sigma + ik\gamma_0 - \varrho) \simeq \Re F(\sigma - \varrho_0), \text{ when } k = 0.$$

Otherwise, we will show that $\mathcal{S}$ is a negligible term.

7.1. **Analysis of the principal term.**

Lemma 7.1.

$$D(\sigma - \beta_0) + D(\sigma - 1 + \beta_0) \geq \tilde{F}(\sigma - \beta_0, 0) - \mathfrak{s}_1(\eta),$$

$$\begin{aligned} \textit{with } \mathfrak{s}_1(\eta) = &- \left[1 - \kappa\left(\frac{1}{\delta} + \frac{1}{\sigma_0 - \eta_0 + \delta}\right)\right] h_\theta(0)\eta \\ &- \left[1 - \kappa\left(\frac{1}{\delta^3} + \frac{1}{(\sigma_0 - \eta_0 + \delta)^3}\right)\right] m_\theta \eta^3. \end{aligned}$$

This was established in section 4.3.1 of [6].

Lemma 7.2. *Let α satisfy $0 < \alpha < \frac{\pi}{d_1(\theta)}$. If $0 \leq \gamma_0 \leq \alpha\eta$, then*

$$D(\sigma - \beta_0 - i\gamma_0) + D(\sigma - 1 + \beta_0 - i\gamma_0) \geq \tilde{F}(\sigma - \beta_0, \alpha\eta) - \mathfrak{s}_1'(\eta)$$

$$\begin{aligned} \textit{with } \mathfrak{s}_1'(\eta) = &h_\theta(0)\left(-\frac{\sigma_0 - 1/2}{1 + \alpha^2\eta_0^2} + \frac{\kappa}{\sigma_0 - 1 + \delta} + \frac{\kappa}{\sigma_0 - 1/2 + \delta}\right)\eta \\ &- m_\theta\left(\frac{1}{\sigma_0 - 1/2} + \frac{\kappa}{(\sigma_0 - 1 + \delta)^3} + \frac{\kappa}{(\sigma_0 - 1/2 + \delta)^3}\right)\eta^3. \end{aligned}$$

Proof. Since $\alpha d_1(\theta) < \pi$, then $\cos(\gamma_0 t/\eta)$ is a decreasing function of γ_0 and $\tilde{F}(\sigma - \beta_0, \gamma_0) \geq \tilde{F}(\sigma - \beta_0, \alpha\eta)$. We use (2.22) to estimate $\tilde{F}(\sigma - 1 + \beta_0, \gamma_0)$, $\tilde{F}(\sigma - \beta_0 + \delta, \gamma_0)$ and $\tilde{F}(\sigma - 1 + \beta_0 + \delta, \gamma_0)$. □

7.2. **Study of the remainder term.**

We now give a lower bound for $\mathcal{S}$. First, we rewrite the sum by applying the symmetry of the zeros so that both ϱ and $1 - \overline{\varrho}$ appear:

$$\begin{aligned} \mathcal{S} = &\sum_{\beta = 1/2} D\left(\sigma - 1/2 + i(k\gamma_0 - \gamma)\right) \\ &+ \sum_{\beta > 1/2} \Big[D(\sigma - \beta + i(k\gamma_0 - \gamma)) + D(\sigma - 1 + \beta + i(k\gamma_0 - \gamma))\Big], \end{aligned}$$

where the sums are taken over the non-trivial zeros $\beta + i\gamma$ of $L(s, \chi')$. When $|\Im s|$ is large enough, we can approximate $\tilde{F}(s)$ using (2.19):

$$\tilde{F}(s) \simeq f(0)\Re\left(\frac{1}{s}\right). \tag{7.1}$$

As a consequence, we expect that, for $|k\gamma_0 - \gamma|$ large enough, the summand $D(\sigma - \beta + i(k\gamma_0 - \gamma)) + D(\sigma - 1 + \beta + i(k\gamma_0 - \gamma))$ will behave like

$$(7.2)\quad f(0)\Re\Bigg[\left(\frac{1}{\sigma-\beta+i\,(k\gamma_0-\gamma)}+\frac{1}{\sigma-1+\beta+i\,(k\gamma_0-\gamma)}\right) - \kappa\left(\frac{1}{\sigma+\delta-\beta+i\,(k\gamma_0-\gamma)}+\frac{1}{\sigma+\delta-1+\beta+i\,(k\gamma_0-\gamma)}\right)\Bigg].$$

This sort of inequality has previously been studied:

Lemma 7.3 (Stechkin - [11])**.** *If* $\beta \in [\frac{1}{2}, 1]$, $y > 0$, $\sigma > 1$ *and* $\tau = \dfrac{1+\sqrt{1+4\sigma^2}}{2}$, *then*

$$\Re\Big(\frac{1}{\sigma-\beta+iy}+\frac{1}{\sigma-1+\beta+iy}\Big)-\frac{1}{\sqrt{5}}\Re\Big(\frac{1}{\tau-\beta+iy}+\frac{1}{\tau-1+\beta+iy}\Big)\geq 0.$$

At p. 326 of [6], we generalized this to the case where F is the Laplace transform of a smooth function satisfying (7.1) and (2.18) and in particular in the case of the function we chose in section 2.3:

Proposition 7.4. *If* $\beta \in [\frac{1}{2}, \sigma]$, $y > 0$ *and* $\sigma \geq \sigma_0$ *with* $\sigma_0 < 1$, *then*

$$D(\sigma - \beta + iy) + D(\sigma - 1 + \beta + iy) \geq 0,$$

as soon as $0 \leq \kappa \leq \min(\kappa_2(\delta), \kappa_2(\delta))$ *and* $\delta \geq \delta(\theta, r)$, *where* $\delta(\theta, r)$ *is the solution in* $[0, 1]$ *of the equation* $\kappa_2(\delta) = \kappa_3(\delta)$, *with*

$$\kappa_2(\delta) = \frac{(2\sigma_0-1)h_\theta(0) - m_\theta\eta_0^2/(2\sigma_0-1)}{(2\delta+1)h_\theta(0)+\left(\frac{1}{\delta}+\frac{1}{2\sigma_0-1+\delta}\right)m_\theta\eta_0^2},$$

$$\kappa_3(\delta) = \frac{(2\sigma_0-1)h_\theta(0) - m_\theta\eta_0^2/(2\sigma_0-1)}{\left(\frac{1}{\delta}+\frac{1+\delta}{(2\sigma_0-1+\delta)^2}\right)h_\theta(0)+m_\theta\eta_0^2\left(\frac{1}{\delta^3}+\frac{1}{(2\sigma_0-1+\delta)^3}\right)}.$$

We call $\kappa = \kappa(\theta, r, R, q)$ *the corresponding value of* κ_2 *at* δ: $\kappa = \kappa_2(\delta) = \kappa_3(\delta)$.

Remark. The approximation: $\sigma_0 = 1 + \mathcal{O}(\eta_0)$ implies the following approximate values for κ and δ:

$$\kappa_2(\delta) = \frac{1}{1+2\delta} + \mathcal{O}(\eta_0) \text{ and } \kappa_3(\delta) = \left(\frac{1}{\delta}+\frac{1}{1+\delta}\right)^{-1} + \mathcal{O}(\eta_0).$$

Thus δ is roughly the solution of the equation

$$1 + 2\delta = \frac{1}{\delta} + \frac{1}{1+\delta}$$

and we see that $\delta \simeq \frac{\sqrt{5}-1}{2}$, $\kappa \simeq \frac{1}{\sqrt{5}}$ and $\sigma + \delta \simeq \frac{\sqrt{5}+1}{2} \simeq \tau$ are close to Stechkin's results. The proposition 7.4 and the bounds

$$1 - \frac{1}{R\log\left(q\left(k\gamma_0+t_0\right)\right)} \leq \sigma < \beta \leq 1 - \frac{1}{R\log\left(q\max\left(1,|\gamma|\right)\right)}$$

imply that:

$$(7.3)\qquad \mathcal{S} \geq \sum_{\substack{\varrho\in Z(\chi')\\ |\gamma|\geq k\gamma_0+t_0}} \Big[D(\sigma-\beta+i(k\gamma_0-\gamma))+D(\sigma-1+\beta+i(k\gamma_0-\gamma))\Big].$$

To find a lower bound for this sum, we need an explicit estimate for the number of non-trivial zeros of $L(s, \chi)$ with bounded imaginary part:

Lemma 7.5 (McCurley - [8]). *Let $0 < \epsilon \le 1/2$, $T \ge 1$ and χ' be a primitive non-principal character modulo q. Let $N(T,\chi')$ be the number of zeros of $L(s,\chi')$ in the rectangle $0 < \Re s < 1$, $|\Im s| < T$, then:*

$$\left| N(T,\chi') - \frac{T}{\pi}\log\left(\frac{qT}{2\pi e}\right)\right| < C_1\log(qT) + C_2,$$

$C_1 = \frac{1+2\epsilon}{\pi\log 2}$, $C_2 = 0.3058 - 0.268\epsilon + \frac{4\log\zeta(1+\epsilon)}{\log 2} - \frac{2\log\zeta(2+2\epsilon)}{\log 2} + \frac{2}{\pi}\log\zeta\left(3/2+2\epsilon\right).$

Choosing $\epsilon = 1/2$, we obtain a good estimate of N for not too large values of T:

$$N_2(T,q) \le N(T,\chi') \le N_1(T,q),$$

$$N_1(T,q) = \frac{T}{\pi}\log\frac{qT}{2\pi e} + c_1\log(qT) + c_2,\ N_2(T,q) = \frac{T}{\pi}\log\frac{qT}{2\pi e} - c_1\log(qT) - c_2,$$

with $c_1 = 0.91845$, $c_2 = 5.36927$.

Lemma 7.6. *For any primitive non-principal character χ' associated to q, $t \ge 0$ and t_0 a positive integer, we have:*

$$\Sigma(t,t_0,\chi') = \sum_{\substack{\varrho\in Z(\chi')\\ |\gamma|\ge t+t_0}} \frac{1}{(\gamma-t)^2} \le w_1(t_0)\log\left(q(t+1)\right) + w_2(t_0),$$

$$w_1(t_0) = \left(\frac{1}{\pi} + 2c_1\right)\sum_{n\ge t_0} n^{-2},$$

$$w_2(t_0) = \left(\frac{1}{\pi} + 2c_1\right)\sum_{n\ge t_0} n^{-2}\log n + \left(2c_1\log 2 + 2c_2 - \frac{\log\pi}{\pi}\right)\sum_{n\ge t_0} n^{-2}.$$

Proof. We split $[t+t_0, +\infty[$ into intervals of length 1:

$$\Sigma(t,t_0,\chi') = \sum_{n\ge 0}\ \sum_{t+t_0+n\le|\gamma|\le t+t_0+n+1} \frac{1}{(\gamma-t)^2}.$$

Since $|\gamma| \ge t+t_0$, then $(|\gamma|-t)^2 \le (\gamma-t)^2$ and we bound each inner sum:

$$\sum_{t+t_0+n\le|\gamma|\le t+t_0+n+1} \frac{1}{(\gamma-t)^2} \le \frac{N(t+t_0+n+1,\chi') - N(t+t_0+n,\chi')}{(n+t_0)^2}, \tag{7.4}$$

where, according to Lemma 7.5,

$$N(t+t_0+n+1,\chi') - N(t+t_0+n,\chi') \le \left(\frac{1}{\pi}+2c_1\right)\log(t_0+n)$$
$$+ \left(\frac{1}{\pi}+2c_1\right)\log(q(t+1)) + 2c_1\log 2 + 2c_2 - \frac{\log\pi}{\pi}.$$

This combines with (7.4) to provide the announced bound for $\Sigma(t,t_0,\chi')$. □

In order to compute $w_1(t_0)$, we use the identity

$$\sum_{n\ge 0}(t_0+n)^{-2} = \zeta(2) - \sum_{n=1}^{t_0-1} n^{-2}$$

and a similar identity for $w_2(t_0)$.
We use the last lemma to estimate $\eta\Sigma(k\gamma_0,t_0,\chi')$:

$$\eta\Sigma(k\gamma_0,t_0,\chi') \le \mathfrak{s}_0(k,t_0) := \begin{cases} \frac{w_1(t_0)}{r} + \frac{w_2(t_0)}{r\log(q_0Y_0)} & \text{when } k=0,\\ w_1(t_0)\frac{\log(q_0(kY_0+1))}{r\log(q_0Y_0)} + \frac{w_2(t_0)}{r\log(q_0Y_0)} & \text{when } k\ge 1,\end{cases} \tag{7.5}$$

where $Y_0 = T_0$ when $\gamma_0 \ge T_0 \ge 1$ and $Y_0 = 1$ when $\gamma_0 \le 1$. We are now prepared to bound the sum $\mathcal{S}$:

Lemma 7.7. *Let κ and δ be defined as in proposition 7.4. Let t_0 such that $-\frac{1}{t_0^{-2}+1}+\kappa(2\delta+1)\le 0$. Then $\mathcal{S}\ge -\mathfrak{s}_2(k,t_0,\eta)$, with*

$$\mathfrak{s}_2(k,t_0,\eta)=\left[\left(\left(1+\frac{2}{t_0^{-2}+1}+\frac{1}{\eta_0}\left(\kappa(2\delta+1)-\frac{1}{t_0^{-2}+1}\right)\right)h_\theta(0)\right.\right.$$
$$\left.\left.+M(-r/R)\right)\eta+\frac{1+2\kappa}{\sigma_0-1/2}m_\theta\eta^2\right]\mathfrak{s}_0(k,t_0).$$

Proof. We need to find a lower bound for each term D appearing in the sum of (7.3). We use (2.20) and Lemma 2.2 to bound $\tilde{F}(s)$ and obtain

$$\begin{aligned}(7.6)\quad \mathcal{S}\ge \eta h_\theta(0)\sum_{\substack{\varrho\in Z(\chi')\\ |\gamma|\ge k\gamma_0+t_0}}&\Re\Big(\frac{1}{\sigma-\beta+i\,(k\gamma_0-\gamma)}+\frac{1}{\sigma-1+\beta+i\,(k\gamma_0-\gamma)}\\ &-\frac{\kappa}{\sigma-\beta+\delta+i\,(k\gamma_0-\gamma)}-\frac{\kappa}{\sigma-1+\beta+\delta+i\,(k\gamma_0-\gamma)}\Big)\\ &-\sum_{\substack{\varrho\in Z(\chi')\\ |\gamma|\ge k\gamma_0+t_0}}\Big(|H(\sigma-\beta,k\gamma_0-\gamma)|+|H(\sigma-1+\beta,k\gamma_0-\gamma)|\\ &+\kappa\,|H(\sigma-\beta+\delta,k\gamma_0-\gamma)|+\kappa\,|H(\sigma-1+\beta+\delta,k\gamma_0-\gamma)|\Big).\end{aligned}$$

We bound each component

$$\begin{aligned}(7.7)\quad &\Re\Big(\frac{1}{\sigma-\beta+i\,(k\gamma_0-\gamma)}+\frac{1}{\sigma-1+\beta+i\,(k\gamma_0-\gamma)}\\ &-\frac{\kappa}{\sigma-\beta+\delta+i\,(k\gamma_0-\gamma)}-\frac{\kappa}{\sigma-1+\beta+\delta+i\,(k\gamma_0-\gamma)}\Big)\\ &\ge\frac{1}{(k\gamma_0-\gamma)^2}\left(-(1-\sigma)+\frac{2\sigma-1}{t_0^{-2}+1}-\kappa(2\delta+1)\right)\\ &\ge-\frac{\eta}{(k\gamma_0-\gamma)^2}\left(1+\frac{2}{t_0^{-2}+1}+\left(-\frac{1}{t_0^{-2}+1}+\kappa(2\delta+1)\right)\frac{1}{\eta_0}\right),\end{aligned}$$

since for t_0 large enough, $-\frac{1}{t_0^{-2}+1}+\kappa(2\delta+1)$ is negative, $\eta<\eta_0$ and $\omega<1$.
We use lemma 2.2 to treat the second sum. We use (2.21) and the fact that $\frac{\sigma-\beta}{\eta}\ge-\frac{1-\sigma}{\eta}\ge-\frac{r}{R}$ to obtain

$$(7.8)\qquad |H(\sigma-\beta,k\gamma_0-\gamma)|\le\frac{M(-r/R)}{(\gamma-k\gamma_0)^2}\,\eta^2.$$

For $x=\sigma-1+\beta,\sigma-\beta+\delta$ and $\sigma-1+\beta+\delta$, (2.22) implies

$$(7.9)\qquad |H(x,k\gamma_0-\gamma)|\le\frac{m_\theta}{(\sigma_0-1/2)\,(\gamma-k\gamma_0)^2}\,\eta^3.$$

We reinsert the inequalities (7.7), (7.8) and (7.9) in (7.6) and we use (7.5) to bound the factor

$$\eta\sum_{|\gamma|\ge k\gamma_0+t_0}\frac{1}{(k\gamma_0-\gamma)^2}.$$

□

8. Study of $D(\sigma-1+ik\gamma_0)$.

We already studied

$$D(\sigma-1)=\tilde{F}(\sigma-1,0)-\kappa\,\tilde{F}(\sigma-1+\delta,0)$$

in section 4.3 of [6] and established:

Lemma 8.1.
$$D(\sigma-1)\le \tilde F(\sigma-1,0)+\mathfrak{p}_0(\eta),\quad \textit{with}\ \ \mathfrak{p}_0(\eta)=-\frac{h_\theta(0)\kappa}{\delta}\eta+\frac{m_\theta\kappa}{(\sigma_0-1+\delta)^3}\eta^3.$$

We now study $\tilde F(\sigma-1,k\gamma_0)$ when γ_0 is not too small, at least not in comparison to $\eta=1-\beta_0$.

Lemma 8.2. *Let* $k\ge 1$ *and* $\alpha\ge \frac{1}{k}\sqrt{\frac{2rM\,(-r/R)}{(1-\kappa)h_\theta(0)}-\omega_0^2}$. *If* $\alpha\eta<\gamma_0<1$*, then:*
$$D(\sigma-1+ik\gamma_0)\le \frac{(1-\kappa)h_\theta(0)}{2}\,\eta\log q.$$

Proof. It follows from lemma 2.2 that
$$D(\sigma-1+ik\gamma_0)\le \tilde F(\sigma-1,k\gamma_0)\le |H(\sigma-1,k\gamma_0)|\le \frac{M\,(-\omega)}{\omega^2+(k\gamma_0/\eta)^2}$$
and the result follows from $\omega_0\le\omega\le r/R$ and the fact that M is decreasing. □

9. Study of D_1 and D_2.

9.1. **Approximating $\Re\psi$.** We denote $\Delta(x,y)=\Re\psi\Big(\frac{x}{2}+i\frac{y}{2}\Big)-\kappa\ \Re\psi\Big(\frac{x+\delta}{2}+i\frac{y}{2}\Big)$ and
$$r_1(x_0,x_1,y_1)=-\gamma\,(1-\kappa)-2\left(\frac{x_0}{x_1^2+y_1^2}-\kappa\frac{x_1+\delta}{(x_0+\delta)^2+y_1^2}\right)+(1-\kappa)\sum_{n=1}^{l}\frac{1}{n}$$
$$-\sum_{n=1}^{l}\left(\frac{4n+2x_0}{(2n+x_1)^2+y_1^2}-\kappa\frac{4n+2x_1+2\delta}{(2n+x_0+\delta)^2+y_1^2}\right)+\frac{x_1+1}{2l}-\kappa\frac{x_0+\delta}{3l},$$
$$r_2(x_0,x_1,y_1)=\frac{1-\kappa}{2}\log\left(\frac{(x_1+\delta)^2}{y_1^2}+1\right)+\frac{1}{y_1}\arctan\frac{y_1}{x_1}+\frac{\kappa}{y_1}\arctan\frac{y_1}{x_1+\delta},$$
$$r_3(x_0,x_1,y_1)=\frac{1}{3y_1}\left(\frac{1}{x_0}+\frac{\kappa}{x_0+\delta}\right)+\frac{1}{2y_1^2}\left(x_1^2+\kappa(x_1+\delta)^2\right).$$

Lemma 9.1. *Let* $\mathfrak{a}=0$ *or* 1 *and* $T\ge 0$*. Then:*
$$\left|\Re\psi\left(\frac{\mathfrak{a}}{2}+\frac14+i\frac{T}{2}\right)\right|\le U(T)=\log\left(6\,(T+12)\right). \tag{9.1}$$
Let $\delta\in[0,1]$, $\kappa\in[0,x/(x+\delta)]$, $0<y_1^2<l=100$, $0<x_0\le x\le x_1<y_1$*. Then:*
$$\Delta(x,y)\le r_1(x_0,x_1,y_1),\ \ \textit{if}\ \ 0<|y|<y_1, \tag{9.2}$$
$$\Delta(x,y)\le (1-\kappa)\log\frac{|y|}{2}+\min(r_2,r_3)(x_0,x_1,y_1),\ \ \textit{if}\ \ |y|\ge y_1. \tag{9.3}$$

To prove this, we will use

- the fact that $\Re\psi\Big(\frac{x}{2}+i\frac{y}{2}\Big)$ is increasing with y,
- Stirling's formula:
$$\Re\psi\Big(\frac{x}{2}+i\frac{y}{2}\Big)=-\gamma-\frac{2x}{x^2+y^2}+\sum_{n=1}^{+\infty}\left(\frac1n-\frac{2(2n+x)}{(2n+x)^2+y^2}\right), \tag{9.4}$$
- the identity:
$$\Re\psi\left(\frac{x+iy}{2}\right)=\frac12\log\left(\frac{x^2+y^2}{4}\right)-\frac{x}{x^2+y^2}+\Re\int_0^{+\infty}\frac{u-[u]-1/2}{\left(u+\frac{x+iy}{2}\right)^2}du, \tag{9.5}$$

  with the estimate for the integral:
$$\left|\Re\int_0^{+\infty}\frac{u-[u]-1/2}{\left(u+\frac{x+iy}{2}\right)^2}du\right|\le\frac1y\arctan\left(\frac{y}{x}\right).$$

- One deduces from it:

$$\left|\Re\psi\left(\frac{x+iy}{2}\right)-\left(\log\left|\frac{y}{2}\right|-\frac{x}{x^2+y^2}\right)\right|\leq\frac{1}{3x|y|}+\frac{x^2}{2y^2}. \tag{9.6}$$

Proof of (9.1). For $|T|\leq 3/2$, we have that

$$\Re\psi\left(\frac{2\mathfrak{a}+1}{4}+i\frac{T}{2}\right)<\Re\psi\left(\frac{2\mathfrak{a}+1}{4}+i\frac{3}{4}\right).$$

It is negative and then $\left|\Re\psi\left(\frac{2\mathfrak{a}+1}{4}+i\frac{T}{2}\right)\right|=-\Re\psi\left(\frac{2\mathfrak{a}+1}{4}+i\frac{T}{2}\right)$. We use (9.4) and bound this last term with

$$\gamma+\frac{4}{2\mathfrak{a}+1}-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{2\mathfrak{a}+1}{n(2n+\frac{2\mathfrak{a}+1}{2})}\leq\log 72.$$

For $|T|<3/2$, we use (9.6):

$$\left|\Re\psi\left(\frac{1+2\mathfrak{a}}{4}+i\frac{T}{2}\right)\right|\leq\left|\log\frac{|T|}{2}\right|+\frac{1}{3}+\frac{4}{27}+\frac{1}{2}\leq\log\left(6\left(T+12\right)\right),$$

which concludes the proof of (9.1). □

Proof of (9.2). When $0<y<y_1$:

$$\Delta(x,y)\leq\Re\psi\left(\frac{x}{2}+i\frac{y_1}{2}\right)-\kappa\,\Re\psi\left(\frac{x+\delta}{2}\right).$$

We use the formula given by (9.4) for $\Re\psi$ and we truncate the sums after l terms. We bound the sums using the fact that $x_0\leq x\leq x_1$ and that

$$\sum_{n>l}\left(\frac{1}{n}-\frac{2(2n+x)}{(2n+x)^2+y_1^2}\right)\leq\sum_{n>l}\left(\frac{1}{n}-\frac{2}{2n+x+1}\right)\leq\frac{x_1+1}{2l},$$
$$\sum_{n>l}\left(\frac{1}{n}-\frac{2}{2n+x+\delta}\right)\geq\frac{x_0+\delta}{3l}.$$

□

Proof of (9.3). We deduce from (9.5) and (9.6) respectively that

$$\Delta(x,y)\leq(1-\kappa)\log\frac{|y|}{2}-\left(\frac{x}{x^2+y^2}-\kappa\,\frac{x+\delta}{(x+\delta)^2+y^2}\right)+\frac{1-\kappa}{2}\log\left(\frac{(x+\delta)^2}{y^2}+1\right)+\frac{1}{y}\arctan\left(\frac{y}{x}\right)+\frac{\kappa}{y}\arctan\left(\frac{y}{x+\delta}\right) \tag{9.7}$$

and

$$\Delta(x,y)\leq(1-\kappa)\log\frac{|y|}{2}+\frac{1}{3|y|}\left(\frac{1}{x}+\kappa\frac{1}{x+\delta}\right)+\frac{1}{2y^2}\left(x^2+\kappa(x+\delta)^2\right). \tag{9.8}$$

Note that $\kappa\leq\dfrac{x}{x+\delta}$ implies $\dfrac{x}{x^2+y^2}-\kappa\,\dfrac{x+\delta}{(x+\delta)^2+y^2}\geq 0$. Together with $y_1\leq y$ and $0<x_0\leq x\leq x_1<y_1$, (9.7) and (9.8) imply the bounds r_2 and r_3 respectively. □

9.2. **Study of D_1.** Let $Y_0\geq 1$. We introduce

$$\mathfrak{v}_0=-\tfrac{1-\kappa}{2}\log\pi+\tfrac{1}{2}\psi\left(\tfrac{3}{2}\right)-\tfrac{\kappa}{2}\psi\left(\tfrac{\sigma_0+\delta}{2}+1\right),\quad\mathfrak{v}_1(k)=-\tfrac{1-\kappa}{2}\log\pi+\tfrac{r_1(\sigma_0+2,3,k)}{2},$$
$$\mathfrak{v}_2=-\tfrac{1-\kappa}{2}\log\pi+\tfrac{1}{2}\psi(1)-\tfrac{\kappa}{2}\psi\left(\tfrac{\sigma_0+\delta}{2}\right),$$
$$\mathfrak{v}_3(k,Y_0)=\tfrac{1-\kappa}{2}\log\tfrac{k}{2\pi}+\tfrac{\min(r_2,r_3)(\sigma_0,2,kY_0)}{2},\qquad\mathfrak{v}_4(k)=-\tfrac{1-\kappa}{2}\log\pi+\tfrac{r_1(\sigma_0,2,k)}{2}.$$

Lemma 9.2. *Let $\sigma_0 < \sigma \le 1$ and $k \ge 1$, then*

$$D_1(\sigma) \le \mathfrak{v}_0, \tag{9.9}$$

$$D_1(\sigma + ik\gamma_0) \le \mathfrak{v}_1(k) \text{ if } 0 \le \gamma_0 < 1, \tag{9.10}$$

$$D_1(\sigma, \chi_{(k)}) \le \frac{1-\kappa}{2}\log q_k + \mathfrak{v}_2, \tag{9.11}$$

$$D_1(\sigma + ik\gamma_0, \chi_{(k)}) \le \frac{1-\kappa}{2}\log(q_k\gamma_0) + \mathfrak{v}_3(k, Y_0), \text{ if } \gamma_0 \ge Y_0 \ge 1, \tag{9.12}$$

$$D_1(\sigma + ik\gamma_0, \chi_{(k)}) \le \frac{1-\kappa}{2}\log q_k + \mathfrak{v}_4(k), \text{ if } 0 < \gamma_0 < 1. \tag{9.13}$$

Proof. Since ψ is an increasing function of the real variable:

$$D_1(\sigma) \le -\frac{1-\kappa}{2}\log\pi + \frac{1}{2}\psi\left(\frac{3}{2}\right) - \frac{\kappa}{2}\psi\Big(\frac{\sigma_0+\delta}{2}+1\Big)$$

$$\text{and } D_1(\sigma, \chi_{(k)}) \le \frac{1-\kappa}{2}\log q_k - \frac{1-\kappa}{2}\log\pi + \frac{1}{2}\psi(1) - \frac{\kappa}{2}\psi\Big(\frac{\sigma_0+\delta}{2}\Big).$$

This gives (9.9) and (9.11) respectively.
Now, let $k \ge 1$. We obtain (9.10) by using (9.2) with $x_0 = \sigma_0 + 2$, $x_1 = 3$ and $y = k\gamma_0 \ge y_1 = kY_0$. We obtain (9.12) and (9.13) by using respectively $x_0 = \sigma_0$, $x_1 = 2$, $y = k\gamma_0 \ge y_1 = kY_0$ in (9.2) and $x_0 = \sigma_0$, $x_1 = 2$, $y = k\gamma_0 \le y_1 = k$ in (9.3). ☐

9.3. **Study of D_2.** We define

$$\mathfrak{w}_0(\eta) = \eta^3 m_\theta\left(\tfrac{1}{\sigma_0^3} + \tfrac{\kappa}{(\sigma_0+\delta)^3}\right) + \eta^2\,\tfrac{m_\theta p_1}{r\log(q_0)\pi\sigma_0}\left(\tfrac{1}{\sigma_0^2} + \tfrac{\kappa}{(\sigma_0+\delta)^2} + 1 + \kappa\right),$$
$$\mathfrak{w}_1(\eta) = \tfrac{\eta^3 m_\theta}{2}\left(\tfrac{1}{\sigma_0^3} + \tfrac{\kappa}{(\sigma_0+\delta)^3}\right) + \tfrac{\eta^2 m_\theta p_1}{r\log(q_0)\pi\sigma_0}\left(\tfrac{1}{\sigma_0^2} + \tfrac{\kappa}{(\sigma_0+\delta)^2} + 1 + \kappa\right),$$
$$\mathfrak{w}_2(\eta) = \tfrac{\eta^3 m_\theta}{2}\left(\tfrac{1}{\sigma_0 Y_0^2} + \tfrac{\kappa}{(\sigma_0+\delta)Y_0^2}\right) + \tfrac{\eta^2 m_\theta p_2}{r\log(q_0)\pi\sigma_0}\left(\tfrac{1}{\sigma_0^2} + \tfrac{\kappa}{(\sigma_0+\delta)^2} + 1 + \kappa\right).$$

Lemma 9.3. *Let $k \ge 1$ and $Y_0 \ge 1$.*

$$D_2(\sigma) \text{ and } D_2(\sigma + ik\gamma_0) \le \mathfrak{w}_0(\eta) \text{ when } 0 \le \gamma_0 < 1, \tag{9.14}$$

$$D_2(\sigma, \chi_{(k)}) \text{ and } D_2(\sigma + ik\gamma_0, \chi_{(k)}) \le \mathfrak{w}_1(\eta) \text{ when } 0 \le \gamma_0 < 1, \tag{9.15}$$

$$D_2(\sigma + ik\gamma_0, \chi_{(k)}) \le \mathfrak{w}_2(\eta) \text{ when } \gamma_0 \ge Y_0 \ge 1. \tag{9.16}$$

Proof. We use (2.22) to bound $|H|$. Then

$$|H(\sigma, k\gamma_0) - \kappa H(\sigma+\delta, k\gamma_0)| \le m_\theta\eta^3\left(\frac{1}{\sigma_0(\sigma_0^2 + (k\gamma_0)^2)} + \frac{\kappa}{(\sigma_0+\delta)((\sigma_0+\delta)^2 + (k\gamma_0)^2)}\right),$$

We have that, when $k = 0$ or ($k \ge 1$ and $0 \le \gamma_0 < 1$):

$$|H(\sigma, k\gamma_0) - \kappa H(\sigma+\delta, k\gamma_0)| \le m_\theta\eta^3\left(\frac{1}{\sigma_0^3} + \frac{\kappa}{(\sigma_0+\delta)^3}\right), \tag{9.17}$$

and when $\gamma_0 \ge Y_0 \ge 1$:

$$|H(\sigma, k\gamma_0) - \kappa H(\sigma+\delta, k\gamma_0)| \le m_\theta\eta^3\left(\frac{1}{\sigma_0(kY_0)^2} + \frac{\kappa}{(\sigma_0+\delta)(kY_0)^2}\right). \tag{9.18}$$

For the term in the integral, we have

$$\Big|H\Big(x_0, T - k\gamma_0\Big)\Big| \le \frac{m_\theta\eta^3}{x_0(x_0^2 + (T-k\gamma_0)^2)} \quad \text{with} \quad x_0 = \sigma - \frac{1}{2}, \sigma - \frac{1}{2} + \delta.$$

We use (9.1) to bound the ψ-term and we obtain:

$$\frac{1}{2\pi}\int_{\mathbb{R}}\left|\Re\psi\left(\frac{2\mathfrak{a}+1}{4} + i\frac{T}{2}\right)\right| |H(x, T-k\gamma_0)|\,dT \le \eta^3\,\frac{m_\theta}{2\pi x}\int_{\mathbb{R}}\frac{U(T)}{x^2 + (T-k\gamma_0)^2}dT \le \eta^3\,\frac{m_\theta}{\pi x}\int_0^{+\infty}\frac{U(T)}{x^2 + (T-k\gamma_0)^2}dT.$$

The second follows from U being even and increasing with the positive reals. We now prove that

$$I := \int_0^{+\infty} \frac{U(T)}{x^2 + (T - k\gamma_0)^2} dT = \mathcal{O}\left(\frac{1}{\eta}\right)$$

We study seperately the cases when γ_0 is small or not.

- When $0 \le \gamma_0 < 1$, then $(T - k\gamma_0)^2 \ge T^2$ and we obtain

$$I \le \int_0^1 \frac{U(T+4)}{x^2} dT + \int_1^{+\infty} \frac{U(T+4)}{T^2} dT \le \frac{p_1}{r\log(q_0)\eta}\left(\frac{1}{x^2} + 1\right),$$

  where $p_1 := 4.803 \ge \int_1^{+\infty} \frac{U(T+4)}{T^2} dT$. Together with (9.17), it gives (9.14) and (9.15).
- When $\gamma_0 \ge Y_0 \ge 1$, then we bound the integral with

$$\int_0^{k\gamma_0 - 1} \frac{U(T)}{(T - k\gamma_0)^2} dT + \int_{k\gamma_0+1}^{+\infty} \frac{U(T)}{(T - k\gamma_0)^2} dT + \frac{1}{x^2}\int_{k\gamma_0-1}^{k\gamma_0+1} U(T) dT.$$

  We compute each term and find that the integrals $\int_0^{k\gamma_0-1} + \int_{k\gamma_0+1}^{+\infty}$ and $\int_{k\gamma_0-1}^{k\gamma_0+1}$ are both bounded with $2\log(6(k\gamma_0+13))$. We obtain:

$$I \le \frac{1}{\eta}\left(1 + \frac{1}{x^2}\right)\frac{p_2}{r},$$

  where $p_2 := 2.404 \ge \frac{2\log(6(k\gamma_0+13))}{\log(q\gamma_0)}$. Together with (9.18), it gives (9.16).

$\square$

10. Some complementary proofs

10.1. Proof of inequality (2.8). We define

$$c_p(\sigma) = c_p(\sigma, \kappa, \delta) := \frac{1}{p^\sigma - 1} - \frac{\kappa}{p^{\sigma+\delta} - 1}, \tag{10.1}$$

$$C_p(\sigma) := a_0 + \sum_{\substack{2 \le k \le 4 \\ p \nmid q_k}} a_k \left(\frac{1-\kappa}{2} c_p(\sigma)^{-1} - 1\right). \tag{10.2}$$

We assume that the a_i's take the values listed in section 2.2.

Lemma 10.1. *Let $0.25 < \kappa < 0.45$, $\delta < 1$ and $\sigma \ge \sigma_0 > 1 - \frac{1}{5\log 12}$. If*

$$(\log 2) h_\theta(0)\left(a_0 c_2(1) + \sum_{2 \le k \le 4} a_k\left(\frac{1-\kappa}{2} - c_2(\sigma_0)\right)\right) + (\log 3)\frac{h_\theta(\eta_0 \log 3)}{3^{\sigma_0}}\left(1 - \frac{\kappa}{3^\delta}\right) C_3(\sigma_0) \ge 0, \tag{10.3}$$

then

$$\frac{1-\kappa}{2} f(0) \sum_{k=1}^4 a_k \log\left(\frac{q}{q_k}\right) + \sum_{k=0}^4 a_k S(\sigma + ik\gamma_0, \chi_{(k)} - \chi^k) \ge 0. \tag{10.4}$$

The condition (10.3) will be true for the values of θ, t_0, r, R chosen in Table 1 to 4.

Proof. We use the same proof as in pp 404–405 of [9]. We input the definition of $S(s,\chi)$ in the sum appearing in (10.4). The left term equals

$$a_0 \sum_{\substack{p|q\\ m\ge1}} \frac{\log p}{p^{m\sigma}} f(m\log p)\left(1-\frac{\kappa}{p^{m\delta}}\right) + \sum_{k=2}^{4} a_k \Bigg(\frac{1-\kappa}{2} f(0)\log\frac{q}{q_k} + \sum_{\substack{p|q,p\nmid q_k\\ m\ge1}} \Re\left(\frac{\chi_{(k)}(p^m)}{p^{imk\gamma_0}}\right)\frac{(\log p)f(m\log p)}{p^{m\sigma}}\left(1-\frac{\kappa}{p^{m\delta}}\right)\Bigg). \tag{10.5}$$

For $k\ge 2$, we use the inequalities:

$$\Re\left(\frac{\chi_{(k)}(p^m)}{p^{imk\gamma_0}}\right)\ge -1,\ \log\frac{q}{q_k}\ge \sum_{p|q,p\nmid q_k}\log p,\ f(0)\ge \frac{1}{c_p(\sigma)}\sum_{m\ge1}\frac{f(m\log p)}{p^{m\sigma}}\left(1-\frac{\kappa}{p^{m\delta}}\right).$$

We deduce from them the lower bound for (10.5):

$$\sum_{p|q}\log p\sum_{m\ge1}\frac{f(m\log p)}{p^{m\sigma}}\left(1-\frac{\kappa}{p^{m\delta}}\right)\Bigg(a_0+\sum_{\substack{2\le k\le 4\\ p\nmid q_k}} a_k\left(\frac{1-\kappa}{2c_p(\sigma)}-1\right)\Bigg). \tag{10.6}$$

It is immediate to see that $c_p(\sigma,\kappa,\delta)$ increases as δ increases and decreases as κ increases. Also it decreases as p or σ increases. The derivatives are

$$\begin{aligned}
\frac{\partial c_p(\sigma)}{\partial p} &= -\frac{p^{\sigma-1}}{(p^\sigma-1)^2(p^{\sigma+\delta}-1)^2}\left(\sigma(p^{\sigma+\delta}-1)^2-\kappa(\sigma+\delta)p^\delta(p^\sigma-1)^2\right)\\
\frac{\partial c_p(\sigma)}{\partial \sigma} &= -\frac{(\log p)p^{\sigma}}{(p^\sigma-1)^2(p^{\sigma+\delta}-1)^2}\left((p^{\sigma+\delta}-1)^2-\kappa p^\delta(p^\sigma-1)^2\right),\\
&\le -\frac{(\log p)p^{\sigma}}{(p^\sigma-1)^2(p^{\sigma+\delta}-1)^2}\left(\sigma(p^{\sigma+\delta}-1)^2-\kappa(\sigma+\delta)p^\delta(p^\sigma-1)^2\right).
\end{aligned}$$

Since $p\ge 2$, $0.25<\kappa<0.45$, $\delta<1$ and $0.9<\sigma<1$, then

$$\sigma(p^{\sigma+\delta}-1)^2-\kappa(\sigma+\delta)p^\delta(p^\sigma-1)^2 \ge 0.9(p^{1.5}-1)^2-0.85p^{0.7}(p-1)^2\ge 0$$

and the negativity of the derivatives follows. We obtain the numerical bounds:

$$\frac{1-\kappa}{2c_p(\sigma,\kappa,\delta)}-1 \ge \frac{1-0.45}{2c_p(1-\frac{1}{5\log 12},0.25,1)}-1\ge \begin{cases}-0.7336 & \text{if } p=2,\\ -0.4890 & \text{if } p=3,\\ -0.0278 & \text{if } p\ge 5,\end{cases} \tag{10.7}$$

which gives, considering all possible values of the a_i's:

$$C_p(\sigma)\ge\begin{cases}-1.6639 & \text{if } p=2,\\ 0.5110 & \text{if } p=3,\\ 0.9722 & \text{if } p\ge 5.\end{cases} \tag{10.8}$$

Note that if q is not a prime power, then (10.6) does not vanish. Also, in the case where the smallest prime dividing q is larger than 3, then it is clear from (10.8) that (10.6) is positive. Considering that q is a composite numbers with divisors $p=2$ and any other prime $p\ge 3$, we have the lower bound

$$\begin{aligned}
&\log 2\sum_{m\ge1}\frac{f(m\log 2)}{2^{m\sigma}}\left(1-\frac{\kappa}{2^{m\delta}}\right)C_2(\sigma)\\
&\qquad + \log 3\sum_{m\ge1}\frac{f(m\log 3)}{3^{m\sigma}}\left(1-\frac{\kappa}{3^{m\delta}}\right)C_3(\sigma)\\
&\qquad \ge \eta(\log 2)h_\theta(0)c_2(\sigma)C_2(\sigma)+\eta(\log 3)\frac{h_\theta(\eta_0\log 3)}{3^\sigma}\left(1-\frac{\kappa}{3^\delta}\right)C_3(\sigma).
\end{aligned} \tag{10.9}$$

We find the bound (10.3) by using the inequalities $f(m\log 2) \le f(0) = \eta h_\theta(0)$, $f(m\log 3) = \eta h_\theta(\eta m \log 3) \ge \eta h_\theta(\eta_0 \log 3)$, $c_2(1) < c_2(\sigma) < c_2(\sigma_0)$, $C_3(\sigma) > C_3(\sigma_0) > 0$ and $C_2(\sigma) = a_0 + \sum_{\substack{2\le k\le 4\\ 2\nmid q_k}} a_k \left(\frac{1-\kappa}{2c_2(\sigma)} - 1\right) < 0.$ □

10.2. **Proof of lemmas 3.1, 4.1, 5.1, 6.1 and 6.2.** We provide here only the proof of lemma 3.1 since the arguments for the other lemmas are just variants of this one. The trigonometric sum is:

$$\Sigma_1 = -\mathfrak{S} + \mathfrak{P} + \mathfrak{V} + \mathfrak{W},$$

where the right terms are respectively given by the sum over the zeros, $D(\sigma - 1 + ik\gamma_0)$, D_1 and D_2:

$$\mathfrak{S} = a_1\left(D(\sigma-\beta_0) + D(\sigma - 1 + \beta_0)\right) + \sum_{k=0}^{4} a_k \sum_{\varrho\in Z(\chi_{(k)})\setminus\{\varrho_0, 1-\overline{\varrho_0}\}} D(\sigma + ik\gamma_0 - \varrho),$$

$$\mathfrak{V} = a_0 f(0) D_1(\sigma) + f(0)\sum_{k=1}^{4} a_k \left(D_1(\sigma + ik\gamma_0, \chi_{(k)}) + \frac{1-\kappa}{2}\log\frac{q}{q_k}\right),$$

$$\mathfrak{P} = a_0 D(\sigma - 1),\ \mathfrak{W} = \sum_{k=0}^{4} a_k\left(D_2(\sigma + ik\gamma_0, \chi_{(k)})\right).$$

We studied them in sections 7, 8 and 9 respectively. More precisely, we use

- lemma 7.1 and lemma 7.7,
- lemma 8.1,
- the inequalities (9.9) when $k = 0$ and (9.12) otherwise,
- lemma 9.3

and they give respectively:

$$\begin{aligned}
&\mathfrak{S} \ge a_1\tilde{F}(\sigma-\beta_0, 0) - \mathfrak{s}(\eta), && \mathfrak{s}(\eta) = a_1\mathfrak{s}_1(\eta) + \textstyle\sum_{k=0}^{4} a_k \mathfrak{s}_2(k, t_0, \eta),\\
&\mathfrak{P} \le a_0\tilde{F}(\sigma-1, 0) + \mathfrak{p}(\eta), && \mathfrak{p}(\eta) = a_0\mathfrak{p}_0(\eta),\\
&\mathfrak{V} \le \tfrac{A(1-\kappa)h_\theta(0)}{2}\,\eta\log(q\max(\gamma_0,1)) + \mathfrak{v}(\eta), && \mathfrak{v}(\eta) = \left(a_0\mathfrak{v}_0 + \textstyle\sum_{k=1}^{4} a_k\mathfrak{v}_3(k, T_0)\right)h_\theta(0)\eta,\\
&\mathfrak{W} \le \mathfrak{w}(\eta), && \mathfrak{w}(\eta) = a_0\mathfrak{w}_0(\eta) + A\mathfrak{w}_2(\eta).
\end{aligned}$$

We deduce that

$$\Sigma_1 \le \frac{A(1-\kappa)}{2} h_\theta(0)\eta\log(q\max(\gamma_0,1)) + a_0\tilde{F}(\sigma-1,0) - a_1\tilde{F}(\sigma-\beta_0,0) + \mathfrak{e}(\eta),$$

where $\mathfrak{e}(\eta)$ is an error term:

$$\mathfrak{e}(\eta) = \mathfrak{s}(\eta) + \mathfrak{p}(\eta) + \mathfrak{v}(\eta) + \mathfrak{w}(\eta).$$

Note that it is a polynomial of degree 3 that can be written

$$\alpha_1(t_0)\eta + \alpha_2(t_0)\eta^2 + \alpha_3\eta^3.$$

We achieve the proof with the following argument:

Lemma 10.2. *If $\alpha_1(t_0) < 0$ and $\mathfrak{e}(\eta_0) \le 0$, then for every $\eta \in [0, \eta_0]$, we have $\mathfrak{e}(\eta) \le 0$.*

Proof. Since $\alpha_1(t_0) < 0$ and $\alpha_3(t_0) > 0$, then $\alpha_1(t_0) + \alpha_2(t_0)\eta + \alpha_3\eta^2$ possesses two distinct real roots of opposite signs. The lemma follows from the observation that, for $\eta \ge 0$, $\mathfrak{e}(\eta)$ is negative up to the positive root and then positive. □

We conclude the proof of Lemma 3.1 by choosing the values of the parameters θ, r, R, t_0 so that the conditions of the lemma 10.2 are satisfied. We justify now that there exist values of t_0 for which $\alpha_1(t_0) < 0$.

10.2.1. *About the choice of* t_0. The term $\alpha_1(t_0)\eta$ comes from the sum over the zeros studied in lemmas 7.1 and 7.7. Its definition is

$$\alpha_1(t_0) = -a_1\left[1-\kappa\left(\frac{1}{\delta}+\frac{1}{\sigma_0-\eta_0+\delta}\right)\right]h_\theta(0)$$
$$+\sum_{k=0}^{4} a_k \mathfrak{s}_0(k,t_0)\left[\left(1+\frac{2}{t_0^{-2}+1}+\frac{1}{\eta_0}\left(\kappa(2\delta+1)-\frac{1}{t_0^{-2}+1}\right)\right)h_\theta(0)+M\left(-\frac{r}{R}\right)\right]$$
$$-a_0\frac{h_\theta(0)\kappa}{\delta}+\left(a_0\mathfrak{v}_0+\sum_{k=1}^{4}a_k\mathfrak{v}_3(k,T_0)\right)h_\theta(0)$$

where

$$\mathfrak{s}_0(k,t_0) \simeq \frac{\log t_0}{t_0 r\log(q_0Y_0)}.$$

We impose in Lemma 7.7 that $\kappa(2\delta+1)-\frac{1}{t_0^{-2}+1} \geq 0$. Then there exists a positive constant C such that

$$\alpha_1(t_0) \simeq -a_1\left[1-\frac{\kappa}{\delta}\right]h_\theta(0)+C\frac{\log t_0}{t_0}.$$

When t_0 is large enough, the left term becomes negative. Therefore, it is possible to find t_0 such that

$$\kappa(2\delta+1)-\frac{1}{t_0^{-2}+1} \geq 0 \text{ and } \alpha_1(t_0) \leq 0. \tag{10.10}$$

On the other hand, R_0 appears to increase with t_0 since it splits in two factors depending on t_0. They contain respectively

- $K(\omega_0,\theta)^{-1}$ that increases with t_0, since $K(\omega,\theta)$ increases with ω which decreases with t_0,
- $1-\kappa$ that decreases very slowly with σ_0 and therefore with t_0.

This observation forces us to take the smallest value for t_0 satisfying (10.10).

10.2.2. *Description of the algorithm computing* $R_0 = R_0(\theta, r, t_0)$.

(1) We vary the value of θ with a precision of 10^{-4} in the interval described in Lemma 2.3.
(2) We set $t_0 = 1$ and vary the values of r with a precision of 10^{-4} in $[5, R]$. We choose the largest r such that $R_0(\theta, r, 1)$ satisfies $r < R_0 \leq r + 10^{-3}$. This gives a provisional r.
(3) We choose t_0 to be the smallest integer value such that (2.24), (2.25), Lemma 10.1 and 10.2 are true.
(4) With this value of t_0, we obtain a final value for r (obtained as described in step (2)).

We repeat this process replacing the value R associated to the old zero-free region, by the value obtained for R_0. We stop when the value of R_0 stabilizes (here we ask for a precision of three digits).
As an example, for $\theta = 1.8422$, $r = 6.052$, $t_0 = 21$, we find $\kappa \geq 0.4351$ and $\delta \leq 0.6221$, $\mathfrak{e}(\eta_0) \leq -1.8026 \leq 0$ and we obtain $R_0 \leq 6.0523$.
Then, we repeat this process until we find $R_0 = 5.8466$.

TABLE 1

Step	θ	t_0	R	r	κ	δ	$\mathfrak{e}(\eta_0)$	ω_0	R_0
Case I.A									
1	1.8422	21	9.6460	6.052	0.4351	0.6221	-1.8026	0.5504	6.0523
2	1.8511	49	6.0523	5.862	0.4296	0.6238	-0.9893	0.8495	5.8622
3	1.8500	48	5.8622	5.847	0.4291	0.6239	-0.0384	0.8748	5.8477
4	1.8499	49	5.8477	5.846	0.4290	0.6240	-0.0001	0.8769	5.8466
Case I.B									
1	1.8311	134	9.6460	6.402	0.4246	0.6262	-0.29	0.3253	6.4029
2	1.8422	229	6.4029	6.286	0.4206	0.6273	-1.02	0.4565	6.2863
3	1.8533	223	6.2863	6.255	0.4212	0.6271	-0.63	0.4638	6.2552
4	1.8622	216	6.2552	6.250	0.4218	0.6278	-0.23	0.4672	6.2504
5	1.8611	217	6.2504	6.249	0.4217	0.6279	-0.39	0.4673	6.2499
Case I.C									
1	1.8311	106	9.6460	6.398	0.4243	0.6263	-0.34	0.3329	6.3992
2	1.8422	183	6.3992	6.281	0.4203	0.6275	-1.35	0.4664	6.2818
3	1.8533	178	6.2818	6.250	0.4208	0.6272	-0.64	0.4741	6.2509
4	1.8622	172	6.2509	6.246	0.4214	0.6270	-0.32	0.4778	6.2463
5	1.8602	173	6.2463	6.245	0.4214	0.6270	-0.48	0.4778	6.2455

TABLE 2

α	θ	t_0	R	r	κ	δ	$\mathfrak{e}(\eta_0)$	ω_0	R_0
Case II.A									
2.6674	1.8937	36	9.6460	6.4020	0.4277	0.6248	-0.11	0.3732	6.4028
Case II.B									
4.2748	1.8714	48	9.6460	6.3050	0.4265	0.6253	-0.18	0.3564	6.3057
Case II.C									
7.9363	1.8500	62	9.6460	6.3100	0.4251	0.6259	-0.80	0.3472	6.3116

TABLE 3

α	θ	t_0	R	r	κ	δ	$\mathfrak{e}(\eta_0)$	ω_0	R_0
Case III.A									
2.6674	1.7501	192	9.6460	6.4030	0.4098	0.6321	-0.05	0.3140	6.4033
Case III.B									
4.2748	1.7000	104	9.6460	6.2980	0.3819	0.6426	-0.10	0.3284	6.2985
Case III.C									
7.9363	1.6500	75	9.6460	6.0680	0.2947	0.6731	-0.29	0.3273	6.0684

TABLE 4

Step	θ	t_0	R	r	κ	δ	$\mathfrak{e}(\eta_0)$	ω_0	R_0
Case IV.A									
1	1.9300	24	1.6130	1.1040	0.3813	0.6410	-0.035	0.5377	1.1049
2	1.9289	115	1.1049	1.0970	0.3676	0.6457	-0.031	0.7135	1.0977
3	1.9289	117	1.0977	1.0970	0.3673	0.6458	-0.029	0.7175	1.0975
Case IV.B									
1	1.9211	13	3.2219	2.0800	0.4182	0.6279	-0.0001	0.5240	2.0809
2	1.9202	79	2.0809	2.0430	0.4095	0.6307	-0.0411	0.7209	2.0432
3	1.9211	82	2.0432	2.0410	0.4091	0.6308	-0.0241	0.7319	2.0412

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