

Problem Solving Period - Friday, September 20

The Pigeonhole Principle

If n pigeons are placed in m pigeonholes, and $n > m$ then at least one pigeonhole contains more than one pigeon.

Generalization If n objects are sorted into m bins then at least one bin holds at least $\lceil \frac{n}{m} \rceil$ objects.

1. Prove that among five different integers there are always three with sum divisible by 3.
2. Show that if there are n people at a party, then two of them know the same number of people (among those present).
3. If 5 points are chosen on a lattice, then at least one pair defines a line segment that contains a lattice point.
4. A lattice point is a point with integer coordinates. Show that if 9 lattice points, no 3 collinear, are taken in 3-dimensional space, then at least one of the segments joining pairs of these points must pass through another lattice point.
5. Prove that any $(n + 1)$ -element subset of $\{1, 2, \dots, 2n\}$ contains two integers that are relatively prime.
6. (A1 2002) Given any five points on a sphere, show that at least four lie in the same closed hemisphere.
7. (a) Show that among any 6 points in a 3×4 rectangle there is a pair of points not more than 5 apart.
(b) Show that among any 9 points in a triangle of area 1, there are 3 points that form a triangle of area at most $1/4$.
(c) Show that given any 9 points in a triangle of area 1, there is a triangle of area at least $1/12$ that does not contain any of those 9 points in its interior.
8. (A2 1954) Consider any five points P_1, P_2, P_3, P_4, P_5 in the interior of a square S of side-length 1. Denote by d_{ij} the distance between the points P_i and P_j . Prove that at least one of the distances d_{ij} is less than $\sqrt{2}/2$.
9. (IMO 1972.) Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.
10. (A1 1978) Let $S = \{1, 4, 7, 10, 13, 16, \dots, 100\}$. Let T be a subset of 20 elements of S . Show that we can find two distinct elements of T with sum 104.