Problem Solving - September 27 - 3:00-4:50 - B650

1. Do the following expressions make any sense and if they do what are they equal to?

$$\sqrt{2+\sqrt{2+\sqrt{2+...}}}, \qquad \frac{1}{1+\frac{1}{1+\frac{1}{1+...}}}$$

2. Find $\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...$

3. Suppose that you draw a 2 by 2 square, then you join the midpoints of its sides to draw another square, then you join the midpoints of that squares sides to draw another square, and so on. Would you need infinitely many pencils to continue this process forever?

4. Sum the infinite series
$$\sum_{i=1}^{\infty}\frac{1}{(3i-2)(3i+1)}$$

5. A sequence is defined by $a_1 = 2$ and $a_n = 3a_{n-1}+1$ find the sum $a_1 + a_2 + ... + a_n$

6. If $\{a_n\}$ is a sequence such that for $n \geq 1$
$$(2-a_n)a_{n+1} = 1$$
prove that $\lim_{n\to\infty} a_n$ exists and is equal to 1.

7. (2000 A1) Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, ...$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$

8. (1994 A1) Suppose that a sequence $a_1, a_2, a_3, ...$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

9. (1993 A2) Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that
$$x_n^2 - x_{n-1}x_{n+1} = 1 \text{ for } n = 1, 2, 3, ...$$
Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

10. (1947 7) Let $f(x)$ be a function such that $f(1) = 1$ and for $x \geq 1$
$$f' = \frac{1}{x^2 + f^2(x)}.$$
Prove that $\lim_{x\to\infty} f(x)$ exists and is less than $1 + \pi/4$

11. (1991 B1) For each integer $n \geq 0$, ;let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?