Problem Solving - October 4 - 3:00-4:50 - B650

The Principle of Mathematical Induction

Let a be an integer and $P(n)$ a statement about n for each integer $n \geq a$. The principle of mathematical induction states that: Suppose the following two conditions are true.

(i) $P(a)$ is true;

(ii) $P(k+1)$ is true whenever $P(k)$ is true.

Then, $P(k)$ is true for all integers $k \geq a$.

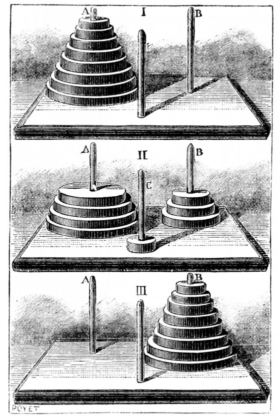
Guidelines

Have fun. There are a lot of problems. Don't try to do all of them. Don't do any you already know how to solve. Work in groups. Don't give up after five minutes. Plug in small numbers. Look for patterns. Draw pictures. Use symmetry. Try cases. Work backwards. If you're stuck, take a break and play a game. Find equivalent versions of the problem. Choose effective notation.

Problems

1. Prove that the following relations hold:
   (a) $1 + 1/\sqrt{2} + 1/\sqrt{3} + ... + 1/\sqrt{n} < 2\sqrt{n}$.
   (b) $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
   (c) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1)n + n(n+1) = \frac{n(n+1)(n+2)}{3}$.
   (d) $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$
2. The Euclidean plane is divided into regions by drawing a finite number of straight lines. Show that it is possible to colour each of these regions either red or blue in such a way that no two adjacent regions have the same colour.
3. If each person, in a group of n people, is a friend of at least half the people in the group, then it is possible to seat the n people in a circle so that everyone sits next to friends only.
4. Let $(x_n)_{n=1}^{\infty}$ be a sequence of integers with $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n = 3, 4, 5, \ldots$.
   (a) Show that $\gcd(f_n, f_{n+1}) = 1$ for all $n \in \mathbb{N}$.
   (b) Show that $f_{n+1} < (7/4)^n$ for all $n \in \mathbb{N}$.
   (c) Show that
   $$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$
   (d) Show that
   $$f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = f_n f_{n+1}.$$

5. There are 3 pegs and n punctured cylindrical disks placed in decreasing order of radius on one of the pegs. The pegs can be moved one at a time and may not be placed on top of a disk of smaller radius. Show that this can be done in $2^n - 1$.



6. Determine a closed formula for the following expression

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\cdots\left(1 - \frac{1}{n^2}\right).$$

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7. How many primes are of the form $101010\cdots101$ base 10?

8. A number n is called **squarefull** if, for any prime $p|n$, $p^2|n$. Show that there are infinitely many consecutive number n and $n+1$ such that both are squarefull.

9. Let $u_1 = 1$ and $u_2 = 2$. For $n > 2$, define the u_n to be the smallest integer k greater than u_{n-1} such that it can be expressed uniquely as the sum of two distinct elements from $\{u_1, u_2, \ldots, u_{n-1}\}$. The first few number in the sequence are $1, 2, 3, 4, 6$. Such numbers are called **Ulam numbers**. Show that there are infinitely many Ulam numbers.

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10. [Putnam 1992 A2] Let $(x_n)_{n\geq 0}$ be a sequence of non-zero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove that there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$.

11. [Putnam 1993 A1] Suppose that a sequence $(a_n)_{n=1}^{\infty}$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

12. [Putnam 1996 B1] Define a **selfish** set to a be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are **minimal** selfish sets, that is, selfish sets none of whose proper subsets are selfish.

13. [Putnam 1959 Morning 1] Let n be a positive integer. Prove that $x^n - (1/x^n)$ is expressible as a polynomial in $x - (1/x)$ with real coefficients if and only if n is odd.