Problem Solving - October 11 - 3:00-4:50 - B650

Guidelines

Have fun. There are a lot of problems. Don't try to do all of them. Don't do any you already know how to solve. Work in groups. Don't give up after five minutes. Plug in small numbers. Look for patterns. Draw pictures. Use symmetry. Try cases. Work backwards. If you're stuck, take a break and play a game. Find equivalent versions of the problem. Choose effective notation.

Problems

1. In a 24-hour period starting at 12:15 AM, how many times on a perfectly functioning standard analogue clock will the minute cross the hour hand? What happens to you answer when we work with an analogue clock that has n hours on its face and we count over $2n$ hour time span? For example, in a 20 hour day with a clock that has ten hours, what should your answer be starting from at a quarter after 10:00 AM (equivalent to noon now)?

2. Show that

$$\begin{aligned}
\sin(2x) &= 2\sin x\cos x\\
\cos(2x) &= (\cos x)^2 - (\sin x)^2\\
\sin(3x) &= -(\sin x)^3 + 3(\cos x)^2\sin x\\
\cos(3x) &= (\cos x)^3 - 3(\sin x)^2\sin x.
\end{aligned}$$

   [A lot of familiarity with trigonometry or a little familiarly with complex number will be very helpful.]

3. How many queens can you put on a chessboard in such a way that none can take another? How many different setups like this exist? Do the same question for rooks and separately for bishops.

4. How many consecutive perfect squares (numbers of the form n^2, where $n \in \mathbb{Z}$) are there?

5. While Jana was preparing the Fun with Math session this afternoon, she came across a question that assumed that there is a point on Earth that has equal day equal night all year long. Adam, being the naturally argumentative pain in the butt that he is, immediately questioned this assumption.

   (a) Show that on any given day there is a point on Earth with equal day equal night.

   (b) Resolve the argument between Jana and Adam by proving or disproving that there is a point on Earth such that for any given day, it has equal day equal night.

   You should note for (b), neither Adam nor Jana know the solution.

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6. (Putnam 1985 A1) Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

   (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

   (ii) $A_1 \cap A_2 \cap A_3 = \varnothing$, where $\varnothing$ denotes the empty set. Express the answer in the form $2^a 3^b 5^c 7^d$, where a, b, c and d are non-negative integers.

7. (Putnam 1987 A1) Curves A, B, C and D are defined in the plane as follows:

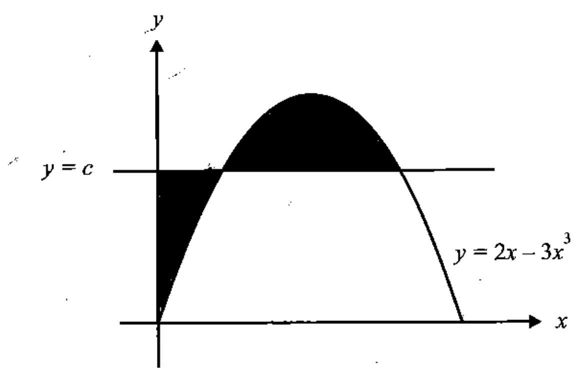
$$\begin{aligned}
A &= \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\} \\
B &= \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\} \\
C &= \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\} \\
D &= \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}.
\end{aligned}$$

Note that $(0,0) \notin A \cup B$, as it leaves those equations in indeterminate. Prove that $A \cap B = C \cap D$.

8. (Putnam 1988 A1) Let R be the region consisting of the points (x,y) of the Cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R, and find its area.

9. (Putnam 1988 B1) A **composite** (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$ with x, y and z positive integers.

10. (Putnam 1993 A1) The horizontal line $y = c$ intersects curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.



11. (Putnam 2001 A1) Consider a set S with a binary operation $*$. That is, for a and $b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.