Problem Solving - October 18 - 3:00-4:50 - B650

Games

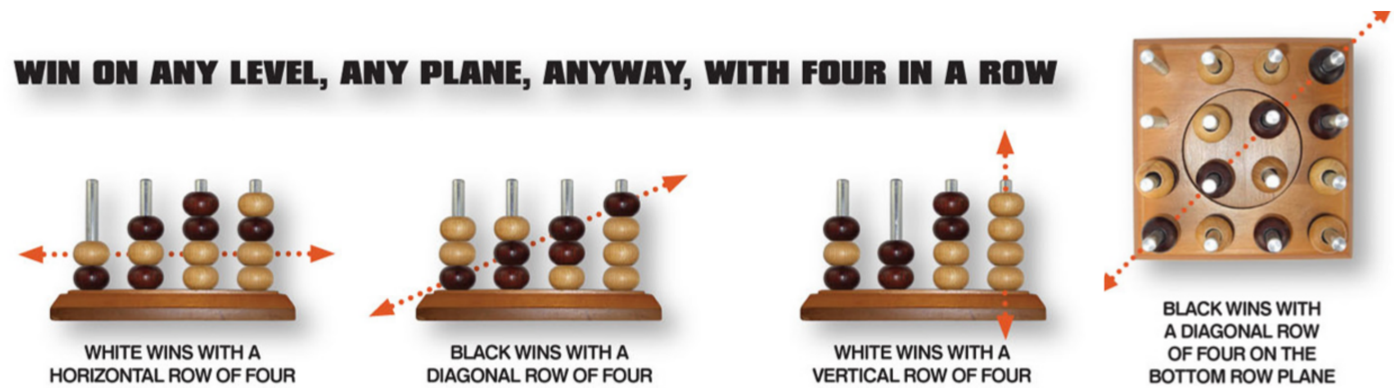
Guidelines

Have fun. There are a lot of problems. Don't try to do all of them. Don't do any you already know how to solve. Work in groups. Don't give up after five minutes. Plug in small numbers. Look for patterns. Draw pictures. Use symmetry. Try cases. Work backwards. If you're stuck, take a break and play a game. Find equivalent versions of the problem. Choose effective notation.

Problems

1. You are playing a game show with the following rules: there are three closed doors in front of you. Behind two doors, there are goats, and behind the remaining door, there is a car. You want to win the car. Once you choose a door, the host will open one of the remaining doors (always revealing a goat) and offer you the option of switching your choice. Should you switch your choice?

2. The game of quadrago (which is in the box at the front of the room) is a three-dimensional connect four game with a twist. You win if you are the first player to have 4 pieces in a row, horizontally, vertically, or diagonally. In each turn you must place a piece, and you have the option of twisting the four columns in the center.



   Suppose, after two turns, all four middle pegs have exactly one piece and the pieces form two horizontal lines of the same colour. Show that the first player can win in under 7 moves.

3. Two players alternately draw diagonals between vertices of a regular polygon. They may connect two vertices if they are non-adjacent and if a diagonal formed does not cross any previous diagonal. Who wins playing on a pentagon? hexagon? 1200-gon? When does player one win and when does player two win?

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4. The game of hex is played on a rhombus with the north-south side painted red and the other two side painted blue. The playing surface is the divided into a hexagonal honeycomb. Alternating, players, take turns painting any (honey)comb red or blue. The first player to connect the opposite sides corresponding to their colour wins. Show that this game cannot end in a tie. (Harder problem) Show that if there is an advantageous strategy, it belongs to the first player.

5. In the game of NIM, N tiles are progressively alternately removed by two players. The player who removes the last tile wins. However, the allowable moves are restricted to a particular set of positive integers M. Determine the values of N for which the first player can win assuming both players play perfectly, where M is the following set:

   (a) $M = \{1, 2\}$;

   (b) $M = \{1, 2, 3\}$;

   (c) $M = \{1, 2, 3, \dots, k\}$;

   (d) $M = \{1, 3\}$;

   (e) (Hard) $M = \{1, 2, 4, 8, \dots\} = \{2^k : k \in \mathbb{N} \cup \{0\}\}$;

   (f) (Harder) $M = \{1, 3, 8\}$;

6. The game of set (which is in the box) is a pattern recognition game. Every card has four characteristics: colour (green, red, or purple), number (1,2, or 3), shading (empty, full, or interlaced), or shape (diamond, oval, or squiggle). A set is a collection of three cards such that, for each of the four characteristics, the characteristic is either all the same or all different. There are examples in the explanation book provided with the game. How many cards would you need in order to guarantee that a set exists? (Harder problem) What is the probability that a collection of n cards has a set?

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7. Two players play a game on a chessboard in which the first player places a king on any empty square of their choosing. Then, starting with the second player, they alternately move the king onto a square on which it has not been already placed until this is not possible. The player who moves last wins. Who has the winning strategy?

8. (Putnam 1993 B2) Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players, A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

9. (IMO 2012 3) The liars guessing game is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

   At the start of the game, A chooses two integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows. Each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in a previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wishes; the only restriction is that, among any $k + 1$ consecutive answers, at least one answer must be truthful.

   After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

   (a) If $n \geq 2^k$, then B can guarantee a win.

   (b) For all sufficiently large k, there exists an integer $n \geq 1.99^k$ such that B cannot guarantee a win. (This is much harder than part (a))