Problem Solving Session

Friday, October 25, 2013

3:00pm-4:50pm in B650

Gee! I'm a tree! (ᵕ̈)

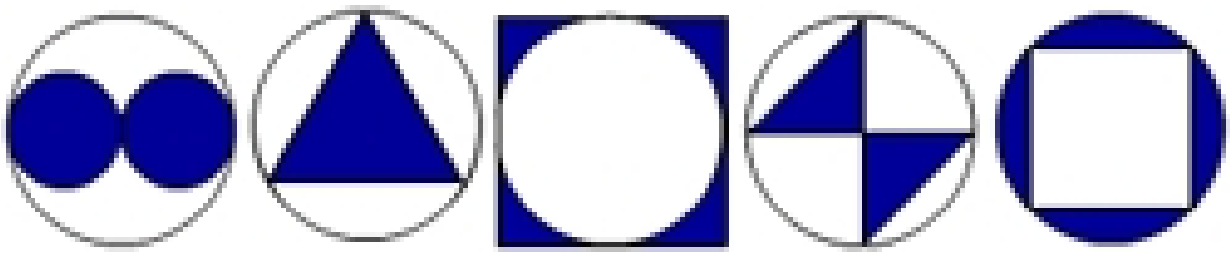
Here are this week's geometry problems:

1. $\triangle ABC$ is an acute triangle whose area is 1. $\triangle DEF$ is created by reflecting $\triangle ABC$ in the line through the midpoints of two of its sides. What is the area of the intersection of $\triangle ABC$ and $\triangle DEF$?

2. A dog is tied to the corner of a shed 3 m wide and 4 m long. The dog can walk around the shed but cannot enter it. If the shed is not next to any obstructions (fences, trees, etc.), what is the area of the region over which the dog can move freely if the length of its leash is…

   (a) …3 m? (b) …4 m? (c) …7 m? (d) …16 m?

3. The largest circle in each figure below has radius 1. In what figure does the shaded region have the largest area? The smallest?



4. A circle of radius 4 is tangent to two lines that are not parallel with each other. A second circle of radius 6 is tangent to the two lines *and* the first circle. A third circle is tangent to the two lines and the second circle. Find the radius of the third circle.

5. Three circles of radius 1 are positioned in a plane so that each one passes through the centers of the other two. What is the area of their intersection?

6. Three circles of radius 1 are positioned in a plane so that each pair of them are externally tangent. A fourth circle is externally tangent with each of the first three circles. What is the area of the fourth circle?

7. E is the midpoint of side BC in rectangle $ABCD$, and F is the midpoint of side CD. The area of $\triangle AEF$ is 3. What is the area of rectangle $ABCD$?

8. Calculate $\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{5}\right)$.

9. Let $ABCD$ be a unit square (its vertices appearing in that order around the square), and let X be a point outside the square such that the distance from X to AC is equal to the distance from X to BD and the distance from A to X is $(\sqrt{2})/2$. What is the distance from C to X?

10. Three circles of radius 1 are positioned in a plane so that no two are tangent but all three intersect at a point O. Show that their other points of intersection lie on a circle of radius 1.

11. Find the area of the region in the xy–plane given by $x^2+y^2 \leq \min\{2x, 2y, 1\}$.

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12. **A1** (December 5, 1998) A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

13. **A1** (December 2, 2006) Find the volume of the region of points (x, y, z) such that

$$\left(x^2+y^2+z^2+8\right)^2 \leq 36\left(x^2+y^2\right).$$

14. **B1** (December 6, 2008) What is the maximum number of rational points that can be on a circle in $\mathbb{R}^2$ whose centre is not a rational point. (A rational point is a point both of whose coordinates are rational numbers.)

15. **A1** (December 1, 2012) Let $d_1, d_2, \ldots d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.