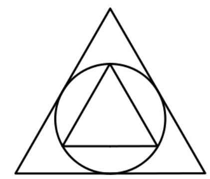
Problem Solving - November 8 - 3:00-4:50 - B650

Guidelines

Have fun. There are a lot of problems. Don't try to do all of them. Don't do any you already know how to solve. Work in groups. Don't give up after five minutes. Plug in small numbers. Look for patterns. Draw pictures. Use symmetry. Try cases. Work backwards. If you're stuck, take a break and play a game. Find equivalent versions of the problem. Choose effective notation.

Problems



1. Inscribe a circle in an equilateral triangle and then inscribe an equilateral triangle in the circle. Using only a picture, find the ratio of the areas of the two triangles.

2. (Larson) Let a, b, c, d be positive numbers. Show that

$$\frac{a^3+b^3+c^3}{a+b+c}+\frac{b^3+c^3+d^3}{b+c+d}+\frac{a^3+c^3+d^3}{a+c+d}+\frac{a^3+b^3+d^3}{a+b+d}\geq a^2+b^2+c^2+d^2.$$

3. Add a joker to a standard deck of cards. Dealing the cards from the top face up, when do you expect the joker to appear? When does the first joker appear when you add N jokers?

4. Jana is driving to work and notices that, on her car's 6-digit odometer, the last 4 digits are palindromic. After one kilometre, the last 5 digits are palindromic. After another kilometre, the middle 4 digits are palindromic. Another kilometre, the all 6 digits are palindromic. What number did Jana start with?

5. (Larson) What is the largest area of a triangle that can be inscribed in a circle? What is the largest area of a rectangle that can be inscribed in a circle?

6. (Larson) Let A, B, C be angle of a triangle. Maximize $\sin A + \sin B + \sin C$.

7. (Larson) Fifteen pool balls are placed in an equilateral triangular rack. The balls are either striped or solid. Show that there are three balls of the same shading that have centres which form an equilateral triangle.

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8. Evaluate the following integrals

(Putnam 1980 A3)

$$\int_0^{\pi/2}\frac{dx}{1+(\tan x)^{\sqrt{2}}}$$

$$\int_0^1 2^{x^2}-1+\sqrt{\log_2(x+1)}\;dx.$$

9. (a) Suppose you are walking home from the university but need to stop at the store. The university, the store and your home form a right isosceles triangle with right angle at store. Halfway between the university and the store, you notice that the sidewalk beyond this point is covered in lava. The store is fine but also surrounded by lava. All along the hypotenuse of the triangle there are lava proof boots available. What is the minimum distance you need to travel in order to get to the store without suffering burns?

   (b) (Putnam 1998 B2) Suppose $0 < b < a$. Determine the minimum perimeter of a triangle with one vertex (a, b), one vertex on the x-axis, and last vertex on the line $y = x$. You may assume that such a triangle exists.