Problem Solving Session

Friday, November 29, 2013

3:00pm-4:50pm in B650

Here are this week's linear algebra problems:

1. In my cupboard is a bowl that contains five brands of multivitamins. Each day, I reach into it and take a couple of complete pills. Before I swallow them, I record the number of units of vitamins A, B, C, and D that I will get from them by referring to this table:

Brand	Vitamin A	Vitamin B	Vitamin C	Vitamin D
1	1	0	1	0
2	1	1	1	1
3	7	7	14	7
4	6	6	12	12
5	9	8	18	19

   In October, I took a total of 564 units of vitamin A, 504 units of B, 1088 units of C, and 1098 units of D. How many of each brand of multivitamin did I consume last month?

2. While preparing practice problems, I created a matrix A with integer entries and computed

$$A^2 = \begin{bmatrix} ? & -6 \\ 2 & 13 \end{bmatrix}.$$

   However, part of my answer got smudged. I also lost the matrix A...whoops! What could the missing entry in A^2 be? How many matrices could I have created?

3. A *Hadamard* matrix of order n is an $n \times n$ matrix A such that every entry is 1 or -1 and $AA^T = nI_n$. Find a Hadamard matrix of order 2. Explain why there is no Hadamard matrix of order 3.

4. Show that if H is Hadamard, then $\begin{bmatrix} H & H \\ -H & H \end{bmatrix}$ is also Hadamard. Use this to find one of order 8.

5. (Putnam 1991 A2) Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

6. Explain how real-valued matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ are like the complex numbers.

7. Calculate the determinant of the following matrix. This is known as the *Vandermonde determinant* of order n.

$$V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^{n-1} \end{vmatrix}$$

8. Show that $\overrightarrow{\mathbf{x}} = \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & a & b \\ a & a^2 & ab \\ b & ab & b^2 \end{bmatrix}$.

   Use this to find all the eigenvalues of A including their algebraic and geometric multiplicities.

9. Let A be a symmetric matrix. Show that if $\overrightarrow{\mathbf{x}}_1$ and $\overrightarrow{\mathbf{x}}_2$ are eigenvectors of A corresponding to different eigenvalues, then $\overrightarrow{\mathbf{x}}_1$ and $\overrightarrow{\mathbf{x}}_2$ are orthogonal.

10. Let $\{M_1, M_2, M_3 \ldots M_k\}$ be a linearly independent set of $n \times 1$ matrices. Show that the set containing the k^2 matrices of the form $M_i M_j^T$ is also linearly independent.

11. (Putnam 1990 B3) Let S be a set of 2×2 integer matrices whose entries a_{ij}

    (a) are all squares of integers and,

    (b) satisfy $a_{ij} \leq 200$.

    Show that if S has more than 50387 $(= 15^4 - 15^2 - 15 + 2)$ elements, then it has two elements that commute.

12. (Putnam 2003 B1) Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

    holds identically?

13. (Putnam 2008 A2) Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if its is zero. Which player has a winning strategy?