Problem Solving Session

Friday, September 26, 2014

2:00pm-2:50pm in C630

Given a predicate $P(n)$, the *Principle of Mathematical Induction* states that if

(i) $P(n_0)$ is true for some integer n_0, and

(ii) for any integer $k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ is true,

then $P(n)$ is true for all integers $n \geq n_0$.

1. Prove: if $n \geq 3$ distinct points on a circle are connected in clockwise order by straight line segments, then the interior angles of the resulting polygon add up to $(n-2)\pi$.

2. Consider the sequence $\{x_n\}_{n=1}^{\infty}$ defined by
$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \sqrt{1+2x_n} \ \text{ for } \ n \geq 1.$$
Prove that $x_n < 4$ for all positive integers n.

3. Suppose $n \geq 1$ chords have been drawn in a circle such that each chord intersects every other chord. Prove that the maximum number of regions into which these chords divide circle is $\left(n^2+n+2\right)/2$.

4. Prove that for all integers $n \geq 1$, $-1+3-5+\cdots+(-1)^n(2n-1) = (-1)^n n$.

5. Prove that $4^{n+1}+5^{2n-1}$ is divisible by 21 for all integers $n \geq 1$.

6. Prove that $n! > 2^n$ for all integers $n \geq 4$.

7. There is an island upon which a tribe resides. The tribe consists of 1000 people, with various eye colours. Yet, their religion forbids them to know their own eye color, or even to discuss the topic; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own (there are no reflective surfaces). If a tribesperson does discover his or her own eye color, then their religion compels them to commit ritual suicide at noon the following day in the village square for all to witness. All the tribespeople are highly logical and devout, and they all know that each other is also highly logical and devout (and they all know that they all know that each other is highly logical and devout, and so forth).

   Of the 1000 islanders, it turns out that 100 of them have blue eyes and 900 of them have brown eyes, although the islanders are not initially aware of these statistics (each of them can of course only see 999 of the 1000 tribespeople).

   One day, a blue-eyed foreigner visits to the island and wins the complete trust of the tribe.

   One evening, he addresses the entire tribe to thank them for their hospitality.

   However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world."

   What effect, if anything, does this faux pas have on the tribe?