Problem Solving - October 24 - 2:00-2:50 - C630

Problems

Combinatorics: The art of counting permutations, combinations, symmetries and the like. Notation: $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$, $n! = n(n-1)\cdots 2 \cdot 1$, $0! = 1$.

1. The following identities are true. Give algebraic or combinatorial proofs for them: for $k, n \in \mathbb{N}$
   (i) $\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k-1}$;
   (ii) (Pascal's Triangle) $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$;
   (iii) (Vandermonde's convolution formula) $\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$;
   (iv) (Subsets of $\{1, 2, \dots, n\}$) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$;
   (v) $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$; $\sum_{k=1}^{n}(-1)^k k\binom{n}{k} = 0$;
   (vi) the number of n-tuples $(x_1, x_2, \dots, x_n)$ with $x_i \in \mathbb{N}$ and $x_1 + x_2 + \cdots + x_n = k$ is $\binom{n+k-1}{n-1}$;
   (vii) $\sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k}$.

2. A mother brings her five children to the Galaxy Cinemas arcade and gives them 30 tokens. In how many ways can she do this if …
   (i) …all of the tokens are the same?
   (ii) …she has 9 Donkey Kong tokens, 10 Mario tokens, and 11 Sonic tokens?
   (iii) …all of the tokens are distinct?

3. (Wilson's theorem) Show that $(n-1)! \equiv -1 \pmod{n}$ if and only if n is a prime number.

4. (Putnam 1985 A1) Determine the number of ordered triples (A_1, A_2, A_3) such that
   (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$;
   (ii) $A_1 \cap A_2 \cap A_3 = \varnothing$.

5. (Putnam 1992 B2) For non-negative integers n and k, define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j}.$$

6. (Putnam 2000 B2) For n, m non-negative integers, prove

$$\frac{\gcd(m,n)}{n}\binom{n}{m} \in \mathbb{Z}.$$