Problem Solving - October 31 - 2:00-2:50 - C630

Problems

Graph theory gives visual representation to objects and their relationships of interest using dots (vertices) and lines (edges (or arrows)). For example, the collection of people you know could be your dots and the lines could be whether two people know each other.

Some terminology: a **simple graph** is given by vertices and edges, where edges are between vertices and there are no multiple edges. All graphs will have a finite number of vertices and a finite number of edges. A few examples will be on the board. In simple graphs, the relationship is usually assumed to be reflexive or we likely don't care about the reverse relationship.

A n**-cycle** is a collection of n distinct vertices $v_1, v_2, \ldots, v_n$ such that v_1 is connected to v_2, v_2 to v_3, and so on, and v_{n-1} is connected to v_n and v_1. A **path** is a string $v_1 e_1 v_2 e_2 v_3 \cdots v_{n-1} e_{n-1} v_n$, where e_i is an edge between v_{i-1} and v_i. A **simple path** is a path in which no vertex is repeated. A **connected graph** is such that, for all pairs of vertices u, v, there exists a path between u and v. A **tree** is a connected graph with no cycles. A **planar graph** is a graph which can be drawn on the plane without any edges crossing. Such a drawing, when it exists, is called a **planar representation**. The number of edges attached to vertex v is called the **degree** of v, denoted $\deg(v)$. An **Eulerian circuit** is a closed path such that every edge is traversed exactly once.

1. At a dinner party people shake hands as they are introduced. Not everyone shakes hands with everyone else (some of them already know each other!). Show that there have to be two people who shake hands the same number of times. Show that the number of people who have shaken hands an odd number of times is even.

2. How many edges does a tree have?

3. Let G be a graph. Suppose all $\deg(v) \geq 2$ for all vertices v in G. Show that G has a cycle. Show that every closed path of odd length has an odd cycle.

4. Show that G has an Eulerian circuit if and only if $\deg(v)$ is even for all vertices v.

5. We say a graph is bipartite if there are two sets of vertices A and B such that A and B are disjoint and $V = A \cup B$, where V is the set of all vertices. Show that a graph is bipartite if and only if the graph does not have an odd cycle.

6. Let G be a connected graph. Show that two simple paths of maximum length must share a vertex in common.

7. Let G be a connected planar graph, and let v, e, f be the number of vertices, edges and faces of G, respectively. Show that $v - e + f = 2$.

8. A platonic solid is a graph such that each vertex has the same degree and each face has the same number of edges. Find all platonic solids.

9. Let G be connected planar graph with v, e, f as above. Suppose $v \geq 3$. Show that $e \leq 3v - 6$. Show that K_5 and $K_{3,3}$ are not planar.

10. Show that every planar graph can be drawn with straight lines.

11. In a party with at least 6 guests, show that there are at least 3 people who all know each other or three people who all do not know each other. (Assume knowing someone is a reflexive relationship.)