Problem Solving Session

Friday, November 14, 2014

2:00pm-2:50pm in C630

Here are this week's linear algebra problems:

1. Find the distance between the lines $x = 1+4t$, $y = 7-t$, $z = 6-t$ and $x = 2t-1$, $y = t$, $z = 5-2t$.

2. A *Hadamard* matrix of order n is an $n \times n$ matrix H such that every entry is 1 or -1 and $HH^T = nI_n$. Find Hadamard matrices of order 2, 4, and 8. Explain why there is no Hadamard matrix of order 3. What is the inverse of a Hadamard matrix?

3. (Putnam 1991 A2) Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

4. Calculate this *Vandermonde determinant* of order n: $V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^{n-1} \end{vmatrix}$

5. Show that $\overrightarrow{\mathbf{x}} = \begin{bmatrix} 1 \\ a \\ b \\ c \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & a & b & c \\ a & a^2 & ab & ac \\ b & ab & b^2 & bc \\ c & ac & bc & c^2 \end{bmatrix}$.

   Use this to find all the eigenvalues and eigenvectors of A.

6. Let A be a symmetric matrix. Show that if $\overrightarrow{\mathbf{x}}_1$ and $\overrightarrow{\mathbf{x}}_2$ are eigenvectors of A corresponding to different eigenvalues, then $\overrightarrow{\mathbf{x}}_1$ and $\overrightarrow{\mathbf{x}}_2$ are orthogonal.

7. Let $\{M_1, M_2, M_3 \ldots M_k\}$ be a linearly independent set of $n \times 1$ matrices. Show that the set containing the k^2 matrices of the form $M_i M_j^T$ is also linearly independent.

8. (Putnam 1990 B3) Let S be a set of 2×2 integer matrices whose entries a_{ij}

   (a) are all squares of integers and,

   (b) satisfy $a_{ij} \leq 200$.

   Show that if S has more than 50387 $(= 15^4 - 15^2 - 15 + 2)$ elements, then it has two elements that commute.