On non-normal 4-valent arc transitive dihedrants

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Joint work with István Kovács and Boštjan Kuzman

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An **n-dihedrant** is a Cayley graph of a dihedral group $D_n$. An **n-bicirculant** is a regular $\mathbb{Z}_n$-cover of a dipole.

\[ n\text{-dihedrant} \Rightarrow n\text{-bicirculant} \]

It is often convenient if we consider dihedrants as bicirculants.
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**BC$_n$1, BC$_n$2, BC$_n$3, BC$_n$4**

Bicirculants of valency 4 fall into 4 classes,

![Diagram of bicirculants with 4 classes]

wrt. the number of perfect matchings between the two orbits of $\mathbb{Z}_n$. 
4 valent edge transitive bicirculants

BC$_n$1, BC$_n$3

No such graphs. Kovács, Kuzman, M., Wilson, 2008
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<td><strong>BC(_n4)</strong></td>
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Non-normal arc-transitive $\text{BC}_{n4}$

Table 1: Non-normal 4-valent arc-transitive dihedrants satisfying the bipartition condition.

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<td>III. The non-incidence graph of $PG(2, 2)$. $n = 7$, $S = {b, ba, ba^2, ba^4}$.</td>
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Arc transitive dihedrants val 4, 1-regular


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Lemma

$X \in BC_n4$ non-normal $\Rightarrow X$ not 1-regular.
References

Arc transitive dihedrants val 4, 1-regular


Lemma

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2-arc transitive dihedrants

I. The lexicographic product $C_n[2K_1]$, $n \geq 4$ even, $S = \{b, ba, ba^2, ba^{2+1}\}$ (in picture, $n = 16$).

II. The graph $K_{5,5} - 5K_2$, $n = 5$, $S = \{b, ba, ba^3, ba^7\}$.

III. The non-incidence graph of $PG(2,2)$, $n = 7$, $S = \{b, ba, ba^2, ba^4\}$.

IV. The incidence graph of $PG(2,3)$, $n = 13$, $S = \{b, ba, ba^3, ba^9\}$.

V. A 2-cover of the graph III, $n = 14$, $S = \{b, ba, ba^4, ba^6\}$.

VI. A 3-cover of the graph II, $n = 15$, $S = \{b, ba, ba^3, ba^7\}$.

Table 1: Non-normal 4-valent arc-transitive dihedrants satisfying the bipartition condition.
With $X \in BC_n4$ we associate a certain circulant $Y$ (step two blue graph in figure below). Graphs $X$ are classified by finding all possible graphs $Y$. 
With \( X \in BC_n^4 \) we associate a certain circulant \( Y \) (step two blue graph in figure below). Graphs \( X \) are classified by finding all possible graphs \( Y \).

With $X \in \text{BC}_n 4$ we associate a certain circulant $Y$ (step two blue graph in figure below). Graphs $X$ are classified by finding all possible graphs $Y$.


In order this to work we need to transfer symmetry properties between $X$ and $Y$. 
Lemma

\( X \in BC_n 4 \) non-normal \( \Leftrightarrow \) \( X \) non-normal \( \mathbb{Z}_n \)-cover of \( \text{dip}_4 \).
Lemma

\[ X \in BC_n^4 \text{ non-normal} \iff X \text{ non-normal } \mathbb{Z}_n\text{-cover of } \text{dip}_4. \]

Lemma

\[ X \in BC_n^4 \text{ non-normal} \implies Y \text{ non-normal circulant.} \]
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Lemma

\( X \in \text{BC}_n4 \) non-normal \( \Rightarrow \) \( Y \) non-normal circulant.

\[ \text{Aut}Y \]

\[ \text{D} \quad \text{G=AutX} \]

\[ \varphi \text{ restr. on } Z \]

\[ Z \quad G \]

\[ \tilde{G} \]

- \( \text{Ker}\phi = \text{Ker}\tilde{G} = 1 \). Then \( Z \triangleleft \text{Aut}(Y) \Rightarrow Z \triangleleft \tilde{G} \Rightarrow Z \triangleleft G \)
- \( \text{Ker}\phi = \text{Ker}\tilde{G} \neq 1 \). Then \( X = C_n[2K_1], \ n \geq 4 \) even, \( Y = C_{n/2}[K_2] \).
Why it works? The structure of $Y$

$X = \text{Cay}(D_n, S), \quad S = \{ b, ba^x, ba^y, ba^z \}$

$Y = \text{Cay}(\mathbb{Z}_n, T), \quad T = \{ a^{\pm x}, a^{\pm y}, a^{\pm z}, a^{\pm (x-y)}, a^{\pm (y-z)}, a^{\pm (x-z)} \}$

might not be arc transitive. However: it is an edge-disjoint union of arc transitive circulants (of which at least one of them is connected).
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**Lemma**

Either

- \( Y \) connected arc transitive, non normal, and
  \[ T = \{a^{\pm x}, a^{\pm y}, a^{\pm z}, a^{\pm(x-y)}, a^{\pm(y-z)}, a^{\pm(x-z)}\} \]

or

- \( Y = Y_1 + Y_2 \), where \( Y_2 \) is connected, arc transitive, non-normal, and
  \[ T_1 = \{a^{\pm x}, a^{\pm(y-z)}\}, \quad T_2 = \{a^{\pm y}, a^{\pm z}, a^{\pm(x-y)}, a^{\pm(x-z)}\} \]
The structure of $Y$, uniformity index of $X$

For $e \in Y$, let $r(e)$ be the number of 3-cycles in $X \cup X^2$ containing $e$. 
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**Lemma**

- If $Y$ is arc transitive, then
  \[ r(e) = \frac{12}{|T|} \text{ for each } e \in E(Y). \]

- If $Y = Y_1 + Y_2$ is an edge disjoint union of two arc transitive graphs, then
  \[ r(e) = \begin{cases} 
    \frac{4}{|T_1|}, & \text{for each } e \in E(Y_1) \\
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$X$ is $k$-uniform if $r(e) = k$ for all $e \in E(Y)$. The parameter $k$ is the uniformity index. If $Y = Y_1 + Y_2$, then $Y_1$ is $k_1$-uniform and $Y_2$ is $k_2$-uniform. Possibly, $k_1 = k_2 = k$. 
Then $Y = Y_1 + Y_2$. Since $|T_1|k_1 = 4$, $|T_2|k_2 = 8 \Rightarrow k_1, k_2 \in \{1, 2, 4\}$. 

By a result of Baik, Feng, Sim, Xu, the graph $Y_2$ is one of $K_5$, $K_5$, $K_2$, and $C_n/2[K_1]$ for $n \geq 6$ even. None of these can appear as $Y_2$. 

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Then \( Y = Y_1 + Y_2 \). Since \( |T_1|k_1 = 4, \ |T_2|k_2 = 8 \ \Rightarrow k_1, k_2 \in \{1, 2, 4\} \).

Lemma

If \( X \) is non-uniform, then \( X = C_n[2K_1], \ n \geq 4 \) even.
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**Lemma**

If $X$ is non-uniform, then $X = C_n[2K_1]$, $n \geq 4$ even.

- $k_1 = 4$. Then elements in $T_1 = \{a^{\pm x}, a^{\pm(y-z)}\}$ coincide. Hence $X = C_n[2K_1]$. 

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- $k_1 = 1$. Then $T_1 = \{a^{\pm(y+z)}, a^{\pm(y-z)}\}$ and $T_2 = \{a^{\pm y}, a^{\pm z}\}$. Now $Y_2$ is a connected 4-valent arc transitive circulant, and non-normal.

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None of these can appear as \( Y_2 \).

Lemma

If $X$ is $k$-uniform, $k > 1$, then $k \leq 4$, and

- $k = 4$ and $X = K_{4,4}$.
- $k = 3$ and $X = K_{5,5} - 5K_2$.
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These graphs are constructed using elementary combinatorial arguments.
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These graphs are constructed using elementary combinatorial arguments.
We use quotienting by the action of some cyclic subgroup $\bar{J} \leq \bar{Z}$ such that

As a quotient we obtain either a cycle, or a smaller 4-valent graph. In the latter case, $X \rightarrow X/\bar{J}$ is a regular cyclic covering projection. Moreover, $X/\bar{J}$ is an arc transitive dihedrant, and non-normal. The uniformity index for the small dihedrant is $> 1$. So $X$ is a regular cyclic cover of $C_m\langle 2K_1, K_4, 4, K_5, 5 \rangle - 5K_2$, non-inc. $PG(2, 2)$.
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How to find \( \bar{J} \)?

It is enough to find \( J < Z < \text{Aut}(Y) \) such that the orbits of \( J \) are blocks of imprimitivity for \( \text{Aut}(Y) \).

Since \( X \neq C_n \), we have a monomorphism of \( \bar{G} \rightarrow \text{Aut}(Y) \), and we can transfer the action of \( J \) back to an action of \( \bar{J} \) on \( X \).
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- \( X/\bar{J} \) is an arc transitive dihedrant, and non-normal.
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"X is 1-uniform"
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Y or \((Y_2)\) is a connected, arc transitive, non-normal circulant. Hence the connection set \(T\) (or \(T_2\)) has restricted structure, by a result of Kovács.


(A) There exists $1 < E < Z$ such that $T$ (or $T_2$) is a union of $E$-cosets,

(B) There exists a coprime decomposition $Z = E \times F$, $|E| > 3$, such that $T$ (or $T_2$) is of the form $E \# T'$, for some $T' < F$. 
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In (A) and (B), cosets of $E$ and $F$ are blocks of imprimitivity for $\text{Aut}(Y)$. 
1 < E < Z, take \( J = E \)

\[ X \rightarrow X/\bar{E} \text{ not a cover} \]
$1 < E < Z$, take $J = E$

$X \rightarrow X/\bar{E}$ not a cover

- $T \cap E \neq \emptyset$. So $Y = Y_1 + Y_2$, and $T_2 \cap E = \emptyset$. Then $T_1 \cap E \neq \emptyset$ and moreover, $T_1 \subset E$.

$|T_2| = 8$ and $|E| \geq 5$ forces $|E| = 8$, $T_2$ is one coset. Also, $E$ is of index 2. So $n = 16$.

Hence $T_1 \subset a^{\pm\{2,4,6,8\}}$ and $T_2 = a^{\pm\{1,3,5,7\}}$. So $x \in \pm\{2,4,6\}$, $y = 1$, $z \in \{3,5,7\}$.

By MAGMA, none of these graphs is arc transitive.
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$X \rightarrow X/\bar{E}$ is a cover

- The quotient graph must be $k$-uniform, $k > 1$. Then $k = |E|$. By direct case analysis, also using MAGMA, we have:
  - $k = 2$. A 2-cover of the graph III. We obtain graph V.
  - $k = 3$. A 3-cover of the graph II. We obtain graph VI.
  - $k = 4$. A 4-cover of $K_{4,4}$. No graphs.
$Z = E \times F$, take $J = E$

$X \to X/E$ not a cover
\[ Z = E \times F, \text{ take } J = E \]

\[ X \to X/\bar{E} \text{ not a cover} \]

- \( T_2 \cap E \neq \emptyset \). Then \( T_2 = E^\# = \mathbb{Z}_n^\# \), so \( n = 13 \). We obtain graph IV.

- \( T_1 \subset E, \ T_1 \neq T_2 \).

More involved. No other graphs.
\( Z = E \times F \), take \( J = E \)

\[ X \to X/E \] not a cover

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  More involved. No other graphs.

\[ X \to X/E \] is a cover. Consider \( X \to X/F \)
$Z = E \times F$, take $J = E$

$X \to X/\bar{E}$ not a cover

- $T_2 \cap E \neq \emptyset$. Then $T_2 = E^\# = \mathbb{Z}_n^\#$, so $n = 13$. We obtain graph IV.
- $T_1 \subset E$, $T_1 \neq T_2$.
  More involved. No other graphs.

$X \to X/\bar{E}$ is a cover. Consider $X \to X/\bar{F}$

- Then $\bar{E} \triangleleft G$. So $X \to X/\bar{F}$ is not a cover. Otherwise $\bar{F} \triangleleft G$, and so $\bar{Z} = \bar{E} \bar{F} \triangleleft G$, a contradiction. Thus, $T_1 \subset F$. It follows that $X/\bar{E}$ is non-uniform, with parameters $(1, |E| - 1)$, a contradiction.
Non-normal arc-transitive $\text{BC}_{n4}$

Table 1: Non-normal 4-valent arc-transitive dihedrants satisfying the bipartition condition.
Thank you!