Searching for an innovative approach to Hamilton cycles/paths problem in vertex-transitive graphs

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Innovative approach?

Innovation?
Suggestions for innovative approach in the development of Coastal region of Slovenia:

- tourism
- transportation
- high tech
- no borders

...bodimo brez meja ...

... usmeritev v izdelke z visoko dodano vrednostjo ...
whereas countries such as Slovenia market themselves as the "sunny side of the Alps", many communities are denied sunshine from November to February because they are on the wrong side of a mountain.

Rattenberg's 460 inhabitants have unusually high rates of seasonal depression, according to Peter Erhard, the local doctor; he says that more of his patients are reporting sleeplessness, sadness, lethargy and poor self-esteem during winter.

the residents of the village in the shadow of the Tyrol mountain range hope to beat the winter blues with 30 computer-controlled solar reflectors, 2.5m x 2.5m giant mirrors or heliostats that can bounce rays to a target.

they have backed a scheme to use to bounce the rays of the sun around the mountains overlooking Rattenberg and on to their high street.

the initiative, backed by the European Union, could become a model for the whole alpine region.
Innovations in Hamilton cycle/path problem in VTG?

Tourists or rays of sunshine?
1. Lovász question

2. Hamilton cycles in (2,s,3)-Cayley graphs

3. Hamilton cycles in graph covers

4. Hamiltonicity of VTG of order a product of two primes
A graph is **vertex-transitive** if its automorphism group acts transitively on vertices. A vertex-transitive graph is a **Cayley graph** if its automorphism group has a regular subgroup.
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A **Hamilton path** is a spanning path. A **Hamilton cycle** is a spanning cycle.
Lovász question

Does every connected vertex-transitive graph have a Hamilton path?

Lovász problem is, somewhat misleadingly, usually referred to as the Lovász conjecture, presumably in view of the fact that, after all these years, a connected vertex-transitive graph without a Hamilton path is yet to be produced.
Only four connected vertex-transitive graphs (having at least three vertices) not having a Hamilton cycle are known to exist:

- the Petersen graph,
- the Coxeter graph,
- and the two graphs obtained from them by replacing each vertex with a triangle.

All of these are cubic graphs, suggesting perhaps that no attempt to resolve the above problem can bypass a thorough analysis of cubic vertex-transitive graphs.

None of these four graphs is a Cayley graph. This has led to a folklore conjecture that every connected Cayley graph has a Hamilton cycle.
Conjectures/counterconjectures

Thomassen, ’91
There exist only finitely many connected vertex-transitive graphs without a Hamilton cycle.

Babai, ’79
There exist infinitely many such graphs.
The current situation

Hamilton cycles (paths) are known to exist in these cases:

- VTG of order $p$, $4p$, $6p$, $2p^2$, $p^k$ (for $k \leq 4$) (Alspach, Chen, Du, Kutnar, Parsons, Šparl, Zhang, DM, etc.);
- CG of $p$-groups (Witte);
- VTG having groups with a cyclic commutator subgroup of order $p^k$ (Durenberger, Gavlas, Keating, Morris, Morris-Witte, DM, etc.).
- CG $\text{Cay}(G, \{a, b, a^b\})$, where $a$ is an involution (Pak, Radoičić).
- Cubic CG $\text{Cay}(G, S)$, where $S = \{a, b, c\}$ and $a^2 = b^2 = c^2 = 1$ and $ab = ba$ (Cherkassoff, Sjerve).
- Cubic CG $\text{Cay}(G, S)$, where $S = \{a, x\}$ and $x^s = 1$, $a^2 = 1$ and $(ax)^3 = 1$ (Glover, DM).
- and in some other cases.

In short: the problem is still open.
Let $s \geq 3$ and $G = \langle a, x | a^2 = 1, x^s = 1, (ax)^3 = 1, \ldots \rangle$ a group with a $(2, s, 3)$-presentation. Then $\text{Cay}(G, \{a, x, x^{-1}\})$ has

- a Hamilton cycle when $|G|$ is congruent to 2 modulo 4, and
- a cycle of length $|G| - 2$, and also a Hamilton path, when $|G|$ is congruent to 0 modulo 4.
Proof strategy

Based on an embedding of $X = \text{Cay}(G, \{a, x, x^{-1}\})$, onto a corresponding orientable surface with $s$-gonal and hexagonal faces, in which one then looks for a long tree of faces – a tree of faces whose boundary is either a Hamilton cycle in $X$ or a cycle missing two adjacent vertices.
Example: $|G| \equiv 2 \pmod{4}$

$G = S_3 \times \mathbb{Z}_3$ with a $(2, 6, 3)$-presentation

$\langle a, x \mid a^2 = x^6 = (ax)^3 = 1, \ldots \rangle$, where $a = ((12), 0)$ and $x = ((13), 1)$. 

The corresponding hexagon graph.

A Hamilton tree of hexagons.

The corresponding Hamilton cycle in $X$. 

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Searching for an innovative approach
Example: \(|G| \equiv 0 \text{ (mod 4)}\)

\[ G = S_4 \text{ with a } (2, 4, 3)-\text{presentation } \langle a, x \mid a^2 = x^4 = (ax)^3 = 1 \rangle, \]

where \(a = (12)\) and \(x = (1234)\).
Let $X$ be a cyclically 4-edge-connected cubic graph of order $n$, and let $S$ be a maximum cyclically stable subset of $V(X)$. Then $|S| = \lfloor (3n - 2)/2 \rfloor$ and more precisely, the following hold.

(i) If $n \equiv 2 \pmod{4}$ then $|S| = (3n - 2)/4$, and $X[S]$ is a tree and $V(X) \setminus S$ is an independent set of vertices;

(ii) If $n \equiv 0 \pmod{4}$ then $|S| = (3n - 4)/4$, and either $X[S]$ is a tree and $V(X) \setminus S$ induces a graph with a single edge, or $X[S]$ has two components and $V(X) \setminus S$ is an independent set of vertices.
To go from a Hamilton path to a Hamilton cycle in a $(2, s, 3)$-Cayley graph of order 0 (mod 4) three cases can occur:

- $s \equiv 0 \pmod{4}$.
- $s \equiv 2 \pmod{4}$.
- $s$ odd.
Hamiltonicity of $(2, s, 3)$-Cayley graphs, $s \equiv 0 \pmod{4}$

Glover, Kutnar, DM

Let $s \equiv 0 \pmod{4} \geq 4$ be an integer. Then a $(2, s, 3)$-Cayley graph has a Hamilton cycle.
Hamiltonicity of \((2, s, 3)\)-Cayley graphs, \(s \equiv 0 \pmod{4}\)

Glover, Kutnar, DM

Let \(s \equiv 0 \pmod{4} \geq 4\) be an integer. Then a \((2, s, 3)\)-Cayley graph has a Hamilton cycle.

Essential ingredients in the proof

- Method used in the proof of the first result.
- Classification of cubic ATG of girth 6.
- Results on cubic ATG admitting a 1-regular subgroup.
Hamiltonicity of \((2, s, 3)\)-Cayley graphs, \(s \equiv 2 \pmod{4}\)

\(s \equiv 2 \pmod{4}\) requires a different approach, work in progress.
Hamiltonicity of \( (2, s, 3) \)-Cayley graphs, \( s \) odd

- \( \langle x \rangle \) is corefree in \( G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots \rangle \):

A method similar to the method used in \( s \equiv 0 \pmod{4} \) case gives us a Hamilton cycle as a boundary of a Hamilton tree of faces consisting of hexagons and two \( s \)-gons.

- \( \langle x \rangle \) is not corefree in \( G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots \rangle \):

Results about lifts of Hamilton cycles in covers of graphs are needed.
Hamiltonicity of $(2, s, 3)$-Cayley graphs, $s$ odd
A general question

When is it true that a vertex-transitive cover of a hamiltonian vertex-transitive graph is also hamiltonian?

For example, if one could show that a connected regular $\mathbb{Z}_p$-cover, where $p$ is a prime, of a hamiltonian vertex-transitive graph of order $p^k$ is hamiltonian, then an inductive argument would yield existence of Hamilton cycles in vertex-transitive graphs of prime power order.
Based on the classification of vertex-transitive graphs of order $pq$, done independently Praeger, Xu, Wang and by DM, Scapellato (early 90s).

**Almost Theorem (Du, Kutnar, DM)**

A connected vertex-transitive graph of order $pq$ other then the Petersen graph contains a Hamilton cycle.
• \exists \text{ transitive } G \leq \text{Aut}X \text{ with } p\text{-blocks} \\
  \iff X = \text{metacirculant} (\text{Alspach, Parsons, DM, early 80s})

• \exists \text{ transitive } G \leq \text{Aut}X \text{ with } q\text{-blocks} (\text{and no subgroup with } p\text{-blocks}) \\
  \iff X = \text{Fermat graph} (p = 2^{2^s} + 1, \ q \text{ divides } 2^{2^s} - 1, \text{ etc.}) \\
  (\text{DM, '92})

• all transitive subgroups of \text{Aut}X are primitive
## Primitive groups of degree $pq$

Primitive groups of degree $pq$ without imprimitive subgroups and with non-isomorphic generalized orbital graphs

<table>
<thead>
<tr>
<th>$soc\ G$</th>
<th>$(p, q)$</th>
<th>$action$</th>
<th>$comment$</th>
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<tbody>
<tr>
<td>$P\Omega^+(2d, k)$</td>
<td>$(k^d + 1, \frac{k^{d-1} - 1}{k-1})$</td>
<td>singular 1-spaces</td>
<td>$d$ a power of 2</td>
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<tr>
<td>$M_{22}$</td>
<td>$(11, 7)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_7$</td>
<td>$(7, 5)$</td>
<td>triples</td>
<td></td>
</tr>
<tr>
<td>$PSL(2, 61)$</td>
<td>$(61, 31)$</td>
<td>cosets of $A_5$</td>
<td></td>
</tr>
<tr>
<td>$PSL(2, q^2)$</td>
<td>$(\frac{q^2+1}{2}, q)$</td>
<td>cosets of $PGL(2, q)$</td>
<td>$q \geq 5$</td>
</tr>
<tr>
<td>$PSL(2, p)$</td>
<td>$(p, \frac{p+1}{2})$</td>
<td>cosets of $D_{p-1}$</td>
<td>$p \equiv 1 (mod\ 4)$ $p \geq 13$</td>
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Chvátal ideals

Chvátal, ’72

Let $X$ be a graph and let $S_i = \{x \in V(X) | \deg(x) \leq i\}$. Then $X$ has a HC if for each $i < n/2$ either $|S_i| \leq i - 1$ or $|S_{n-i-1}| \leq n - i - 1$. 
Example

\[ S_1 = \emptyset, \; S_2 = \emptyset \text{ and } |S_3| = 3 \]

\[ |S_3| \leq 2 \text{ or } |S_{7-3-1}| = |S_3| \leq 7 - 3 - 1 = 3 \]

\[ \Rightarrow \text{ By Chvátal the graph has a Hamilton cycle.} \]
The strategy consists in:

- first, quotiening the graph with respect to a semiregular automorphism of order $p$,
- second, using Chvátal theorem, finding a Hamilton cycle in the corresponding quotient graph of order $q$,
- third, lifting this cycle to a Hamilton cycle in the original graph.
Example - A graph arising from the action of $PSL(2, 13)$ on the cosets of $D_{12}$

$p = 13$ and $q = 7$
Example - A graph arising from the action of $PSL(2, 13)$ on the cosets of $D_{12}$

$p = 13$ and $q = 7$
Open case

This approach works in all cases with the exception of the case when $G = PSL(2, p)$, $p \equiv 1 \pmod{8}$ and the corresponding graph of order $p(p + 1)/2$ has valency $(p - 1)/4$.

Work in progress linking the existence of Hamilton cycles with certain number-theoretic conditions.
Thank you!

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