On half-arc-transitive metacirculants of valency 4

Primož Šparl

University of Ljubljana, Slovenia

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A half-arc-transitive (HAT) graph is a vertex- and edge- but not arc-transitive graph.
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Since a HAT graph is edge transitive but not arc-transitive, none of its automorphisms can interchange a pair of adjacent vertices.

Hence, with each HAT graph two (paired) oriented graphs are associated in a natural way.

It follows that HAT graphs have even valency (Tutte 1966).
Half-arc-transitive graphs

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- Since a HAT graph is edge transitive but not arc-transitive, none of its automorphisms can interchange a pair of adjacent vertices.
- Hence, with each HAT graph two (paired) oriented graphs are associated in a natural way.
- It follows that HAT graphs have even valency (Tutte 1966).
- The smallest admissible valency for HAT graphs is four.
- This is the class of graphs we want to focus on.
In the last two decades HAT graphs and in particular tetravalent HAT graphs have been studied quite a lot. Many interesting results have been obtained but...
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What about tetravalent HAT graphs?

Even with this restriction there are still numerous questions that need to be answered.
In 1998 Marušič introduced the concept of attachment of alternating cycles.
The alternating cycles in tetravalent HAT graphs.
The radius.
The attachment number.
What is the relationship between these two parameters?
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Which tetravalent HAT graphs have “the nicest” structure with respect to these two parameters?
The alternating cycles and attachment

What is the relationship between these two parameters?
Which tetravalent HAT graphs have “the nicest” structure with respect to these two parameters?
Tightly attached graphs - the two parameters coincide.
In 1998 Marušič classified the odd radius tightly attached tetravalent HAT graphs and in 2008 Šparl classified the even radius graphs.

These graphs are the $X_o(m, n; r)$ and $X_e(m, n; r, t)$ graphs. They are clearly weak metacirculants.
Let $m \geq 1$, $n \geq 2$ and let $X$ be a graph of order $mn$.

An automorphism $\rho \in \text{Aut}X$ is \((m, n)\)-semiregular, if it has $m$ orbits of length $n$. 
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An automorphism $\rho \in \text{Aut}X$ is \textit{(m, n)-semiregular}, if it has $m$ orbits of length $n$.

$X$ is an \textit{(m, n)-weak metacirculant}, if it admits a transitive subgroup of automorphisms generated by two automorphisms $\rho$ and $\sigma$ such that $\rho$ is \textit{(m, n)-semiregular} and $\sigma$ normalizes $\rho$, that is $\sigma^{-1}\rho\sigma = \rho^r$ for some $r \in \mathbb{Z}_n^*$. 
A tetravalent HAT weak metacirculant

The Doyle-Holt graph is a \((3, 9)\)-weak metacirculant.
The four classes of tetravalent HAT weak metacirculants

If $X$ is a tetravalent HAT weak metacirculant where $\rho$ is the above semiregular automorphism, what can we say about the quotient (multi)graph $X_\rho$?
The four classes of tetravalent HAT weak metacirculants

If $X$ is a tetravalent HAT weak metacirculant where $\rho$ is the above semiregular automorphism, what can we say about the quotient (multi)graph $X_\rho$?

Theorem (Marušič, Šparl)

*Every tetravalent HAT weak metacirculant belongs to one (or more) of the following four classes:*
The four classes of tetravalent HAT weak metacirculants

Class I

Class II

Class III

Class IV
The four classes of tetravalent HAT weak metacirculants

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  - Are there any tetravalent HAT weak metacirculants that are not tightly attached?
  - What is the relationship between the four classes?
  - Can we classify each of them?
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Theorem (Marušič, Šparl)

*A connected tetravalent HAT graph is a weak metacirculant of Class I if and only if it is tightly attached.*
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As a consequence, each Class I graph has vertex-stabilizers $\mathbb{Z}_2$. 
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- As a consequence, each Class I graph has vertex-stabilizers $\mathbb{Z}_2$.
- Not all Class I graphs are Cayley graphs though. The smallest such example is $\chi_0(4, 17; 2)$. 
Marušič and Šparl proved that every tetravalent HAT weak metacirculant of Class II is isomorphic to a graph $Y(m, n; r, t)$. 

$Y(m, n; r, t)$
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Some of them are not tightly attached. In fact, a $Y(m, n; r, t)$ graph is tightly attached if and only if
$r - 1 \in \langle t - m - \frac{m(m-1)}{2} (r - 1) \rangle$. The smallest non tightly attached example is $Y(4, 48; 13, 44)$. 
Class II

- Each Class II graph is a Cayley graph of some metacyclic group.
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- If a Class II graph is not tightly attached then the ratio between its radius and attachment number is either 2 or 4.
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- If a Class II graph is not tightly attached then the ratio between its radius and attachment number is either 2 or 4.
- Šparl recently classified Class II and showed that except for the Doyle-Holt graph and its canonical double cover all Class II graphs are also of Class IV.
All Class III graphs can naturally be divided into two subclasses.
The first subclass contains the graphs for which the graph obtained from the quotient graph $X_\rho$ by removing all of the double edges is a cycle.
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- The second contains those for which the obtained graph is a union of two cycles.
- Each member of the first subclass is isomorphic to some $\mathcal{W}_1(m, n; r, t, x)$ and each member of the second subclass is isomorphic to some $\mathcal{W}_2(m, n; r, t, x, s)$.
The first subclass of Class III

- By investigating certain 10-cycles inside the $\mathcal{W}_1(m, n; r, t, x)$ graphs we find that:

  - If $m \equiv 2 \pmod{4}$ then $\mathcal{W}_1(m, n; r, t, x)$ is tightly attached and thus of Class I.
  - If $m \equiv 0 \pmod{4}$ then $\mathcal{W}_1(m, n; r, t, x)$ might or might not be of Class I. However, it is always of Class II.

Thus the first subclass of Class III is contained in the union of Classes I and II.
By investigating certain 10-cycles inside the $\mathcal{W}_1(m, n; r, t, x)$ graphs we find that:

- The vertex stabilizers are $\mathbb{Z}_2$. 
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- Thus the first subclass of Class III is contained in the union of Classes I and II.
The second subclass of Class III

- Again investigating certain $10$-cycles inside the $\mathcal{W}_1(m, n; r, t, x, s)$ graphs we find that:

  - The vertex stabilizers are $\mathbb{Z}_2$.
  - All graphs of this subclass are of Class II.

**Theorem**

Let $X$ be a tetravalent half-arc-transitive weak metacirculant of Class III. Then $X$ belongs to the union of Classes I and II. Moreover, $X$ is a Cayley graph of a metacyclic group and has vertex stabilizers $\mathbb{Z}_2$. 
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**Theorem**

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Recall that, except for \( \gamma(3, 9; 4, 6) \) and \( \gamma(3, 18; 7, 15) \), which are both tightly attached and thus of Class I, all Class II graphs are of Class IV.
Recall that, except for $\mathcal{Y}(3, 9; 4, 6)$ and $\mathcal{Y}(3, 18; 7, 15)$, which are both tightly attached and thus of Class I, all Class II graphs are of Class IV.

**Theorem**

*Every connected tetravalent half-arc-transitive weak metacirculant is contained in the union of Classes I and IV.*

In other words, if a tetravalent HAT weak metacirculant is not tightly attached, then it is of Class IV.
Conclusions

It is also easy to see that each Class IV graph is a Cayley graph of a metacyclic group. Thus if a tetravalent HAT weak metacirculant is not a Cayley graph, then it is tightly attached.
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- **Conjecture:** Every connected tetravalent HAT weak metacirculant has vertex stabilizers $\mathbb{Z}_2$. 
Some statistics

The number of Class I and Class II graphs up to a given order:

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<th>TA even</th>
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