CI-groups with respect to ternary relational structures

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Definition

A \textit{k-ary relational structure} is an ordered pair \(X = (V, E)\), with \(V\) a set and \(E\) a subset of \(V^k\). If \(k = 2\), we get the usual definition of digraph. If \(k = 3\), we say that \(X\) is a \textit{ternary relational structure}.
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If $k = 2$, then we get the definition of a Cayley graph over the group $G$ (the graph $X$ is a Cayley graph over the group $G$ if and only if $\text{Aut}(X)$ contains a regular subgroup isomorphic to $G$).
Note that if $X = (G, E)$, $X' = (G, E^\alpha)$ are $k$-ary relational structures over $G$ (where $\alpha \in \text{Aut}(G)$), then $X \cong X'$. 
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We say that $G$ is a **CI-group** with respect to $k$-ary relational structures if whenever $X$ and $Y$ are $k$-ary rel. structures over $G$, $X$ and $Y$ are isomorphic if and only if they are isomorphic by a group automorphism of $G$. 
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The isomorphism problem for $k$-ary relational structures over a CI-group is “as easy as possible”.
The group $G$ is a CI-group with respect to $k$-ary rel. structures if and only if any two regular subgroups isomorphic to $G$ in $\text{Aut}(X)$ are conjugate in $\text{Aut}(X)$ (for any $k$-ary rel. structure $X$ over $G$).
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Case $k = 2$. A relatively short list of groups is given and it is proved that every CI-group with respect to digraphs lies on this list (although not every group on the list need to be a CI-group with respect to digraphs).
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There is a natural 4-ary relational structure encoding the group multiplication of $G$:

$$E = \{(1, x, y, xy)z \mid x, y, z \in G\}.$$

If $X = (G, E)$, then $\text{Aut}(X) = G \rtimes \text{Aut}(G)$. 
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In particular, \( G \) is abelian.
Case $k = 3$. As motivation to study this problem we remark that if $G$ is a CI-group with respect to ternary relational structures then $G$ is a CI-group with respect to digraphs.
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All “elementary” constructions for proving that a certain group $G$ is not a CI-group with respect to digraphs can be used to prove that it is not a CI-group with respect to ternary relational structures.
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All “elementary” constructions for proving that a certain group $G$ is not a CI-group with respect to digraphs can be used to prove that it is not a CI-group with respect to ternary relational structures.

For example, the following are not CI-groups w.r.t. t.r.s.

$$C_{p^2} \ (p \geq 3), \ C_p \rtimes C_q \ (q \geq 3).$$
Proposition (T.Dobson, P.S.)

If $V$ is an abelian group and $\alpha$ is an automorphism of order $p \neq 2$ acting with no fixed points on $V \setminus \{0\}$, then $V \times C_p$ is not a CI-group w.r.t. ternary relational structures.
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\( C_p \times C_p \) is not a CI-group w.r.t. ternary relational structures (\( p \geq 3 \)).
Theorem
If $G$ is a CI-group w.r.t. ternary relational structures, then $G = U \times V$, where $\gcd(|U|, |V|) = 1$, $U$ is of order $n$, with $\gcd(n, \varphi(n)) = 1$ and $V$ is one of the following:

1. $C_{2}^{d}$, $1 \leq d \leq 4$, $D(m, 2)$ or $D(m, 4)$, where $m$ is odd and $\gcd(nm, \varphi(nm)) = 1$.
2. $C_{4}, Q_{8}$.

Furthermore,

(a) if $V = C_{4}, Q_{8}$ or $D(m, 4)$ and $p \mid n$, then $4 \nmid (p - 1)$,
(b) if $V = C_{2}^{d}$, $d \geq 2$, or $Q_{8}$, then $3 \nmid n$,
(c) if $V = C_{2}^{d}$, $d \geq 3$, then $7 \nmid n$,
(d) if $V = C_{2}^{4}$, then $5 \nmid n$. 
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Computational aspects

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The group $H$ is a CI-group w.r.t. $k$-ary relational structures if and only if any two regular subgroups isomorphic to $H$ in $G$ are conjugate in $G$, for any permutation $p$-group $G = N^{(k)}$, where $N$ is a permutation $p$-group normalizing and containing the right regular representation of $H$. 
Given a graph $\Gamma$ determine all the groups $H$ such that

$$\Gamma \cong \text{Cay}(H, S),$$

for some subset $S$ of $H$. 
Example

Let $\Gamma_d$ be the $d$-dimensional cube.

Note that $\Gamma_d$ is a Cayley graph. If $H$ is an elementary abelian 2-group of rank $d$ and $S$ is a basis for $H$, then $\Gamma_d = \text{Cay}(H, S)$.

Proposition (P.S.)

There exist at least $2^{d^2/64}$ non-isomorphic groups $H$ such that $\Gamma_d = \text{Cay}(H, S)$ for some subset $S$ of $H$. 
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Note that it was proved by L. Pyber that $\text{Sym}(n)$ contains at most $24^{n^2}$ subgroups up to isomorphism.