Imprimitive symmetric graphs with cyclic blocks

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3 Imprimitive symmetric graphs with cyclic blocks
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4 Questions
An arc of a graph is an ordered pair of adjacent vertices.
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\( s \text{-arc} \neq \text{path of length } s \).
Arc and $s$-arc

- An arc of a graph is an ordered pair of adjacent vertices.
- An $s$-arc is a sequence of $s + 1$ vertices such that any two consecutive terms are adjacent and any three consecutive terms are distinct.
- $s$-arc $\neq$ path of length $s$.
- $\text{Arc}(\Gamma) :=$ set of arcs of a graph $\Gamma$. 

Let $\Gamma$ admit a group $G$ as a group of automorphisms.
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That is, for any $\alpha, \beta \in V(\Gamma)$ there exists $g \in G$ permuting $\alpha$ to $\beta$, and for any $(\alpha, \beta), (\gamma, \delta) \in \text{Arc}(\Gamma)$ there exists $g \in G$ mapping $(\alpha, \beta)$ to $(\gamma, \delta)$. 
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$\Gamma$ is called $(G, s)$-arc transitive if $G$ is transitive on $V(\Gamma)$ and the set of $s$-arcs.
Tutte’s 8-cage is 5-arc transitive. It is a cubic graph of girth 8 with minimum order (30 vertices).
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$\text{Cos}(G, H, HgH)$ is connected if and only if $\langle H, g \rangle = G$. 
Let $\Gamma$ be a $G$-symmetric graph.
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A partition $\mathcal{B}$ of $V(\Gamma)$ is $G$-invariant if for any $B \in \mathcal{B}$ and $g \in G$, $B^g := \{\alpha^g : \alpha \in B\} \in \mathcal{B}$, e.g. trivial partitions $\{\{\alpha\} : \alpha \in V(\Gamma)\}$ and $\{V(\Gamma)\}$. 

$\Gamma_{\mathcal{B}}$ := quotient graph w.r.t $\mathcal{B}$.

$\Gamma_{\mathcal{B}}$ is $G$-symmetric.
Imprimitive symmetric graphs

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$\Gamma$ is imprimitive if $V(\Gamma)$ admits a nontrivial $G$-invariant partition $\mathcal{B}$.
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$\Gamma$ is imprimitive if $V(\Gamma)$ admits a nontrivial $G$-invariant partition $\mathcal{B}$.

This occurs if and only if $G_\alpha$ is not a maximal subgroup of $G$, where $G_\alpha := \{g \in G : \alpha^g = \alpha\}$. 

Equivlently, Cos($G,H,HgH$) is imprimitive if and only if $H$ is not a maximal subgroup of $G$. 

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$\Gamma_\mathcal{B} :=$ quotient graph w.r.t $\mathcal{B}$.

$\Gamma_\mathcal{B}$ is $G$-symmetric.
The dodecahedron graph is $A_5$-symmetric and the partition with each part containing antipodal vertices is $A_5$-invariant. The quotient graph is isomorphic to Petersen graph.
Let \((\Gamma, \mathcal{B})\) be an imprimitive \(G\)-symmetric graph.
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Let $(\Gamma, B)$ be an imprimitive $G$-symmetric graph.

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- \(\Gamma[B, C] :=\) bipartite subgraph of \(\Gamma\) induced by \(B \cup C\) with isolates deleted.
- Consider the case \(k = v - 2 \geq 1\).
Case $k = v - 2 \geq 1$
An auxiliary graph when $k = v - 2 \geq 1$

\[ \langle B, C \rangle := \text{set of vertices of } B \text{ not adjacent to any vertex of } C. \]
An auxiliary graph when $k = v - 2 \geq 1$

- $\langle B, C \rangle :=$ set of vertices of $B$ not adjacent to any vertex of $C$.
- $|\langle B, C \rangle| = 2$. 

$\Gamma_B :=$ multigraph with vertex set $B$ and an edge joining the two vertices of $\langle B, C \rangle$ for each $C$ adjacent to $B$ in $\Gamma_B$.

Simple $(\Gamma_B) :=$ underlying simple graph of $\Gamma_B$.

Simple $(\Gamma_B)$ is $G_B$-vertex- and $G_B$-edge-transitive, where $G_B := \{g \in G : Bg = B\}$. 

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- \( \Gamma^B := \text{multigraph with vertex set } B \text{ and an edge joining the two vertices of } \langle B, C \rangle \text{ for each } C \text{ adjacent to } B \text{ in } \Gamma^B \).
- \( \text{Simple}(\Gamma^B) := \text{underlying simple graph of } \Gamma^B \).
- \( \text{Simple}(\Gamma^B) \) is \( G_B \)-vertex- and \( G_B \)-edge-transitive, where 
  \( G_B := \{ g \in G : B^g = B \} \).
Lemma

[Iranmanesh, Praeger, Z] Suppose $k = v - 2 \geq 1$. Then one of the following occurs:

(a) $\Gamma^B$ is connected;

(b) $v$ is even and $\text{Simple}(\Gamma^B)$ is a perfect matching between the vertices of $B$. 
Known results when $k = v - 2 \geq 1$

- $\Gamma_B$ is $(G, 2)$-arc transitive (even if $\Gamma$ is not $(G, 2)$-arc transitive) iff $\Gamma^B$ is simple and $v = 3$ or $\Gamma^B = (v/2) \cdot K_2$ (Iranmanesh, Praeger, Z).
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- We know when $\Gamma_B$ inherits $(G, 2)$-arc transitivity from $\Gamma$;
Known results when $k = v - 2 \geq 1$

- $\Gamma_\mathcal{B}$ is ($G, 2$)-arc transitive (even if $\Gamma$ is not ($G, 2$)-arc transitive) iff $\Gamma^B$ is simple and $v = 3$ or $\Gamma^B = (v/2) \cdot K_2$ (Iranmanesh, Praeger, Z).

- We know when $\Gamma_\mathcal{B}$ inherits ($G, 2$)-arc transitivity from $\Gamma$;

- and some information about $\Gamma$ and $\Gamma_\mathcal{B}$ in this case (Iranmanesh, Praeger, Z).
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Known results when \( k = v - 2 \geq 1 \)

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- But we know nothing about \( \Gamma \) and \( \Gamma_B \) when \( \Gamma^B \) is connected (except \( v = 3 \)).

- Let us try the simplest case, namely \( \text{Simple}(\Gamma^B) \) has valency 2.
The case when $\text{Simple}(\Gamma^B)$ has valency 2
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Theorem

[Li, Praeger, Z] Suppose $k = v - 2 \geq 1$, $\Gamma_B$ is connected and $\text{Simple}(\Gamma^B)$ has valency 2. Then $\text{Simple}(\Gamma^B) = C_v$, $\Gamma_B$ is of valency $mv$ (where $m$ is the multiplicity of each edge of $\Gamma^B$), and one of the following occurs.

(a) $v = 3$ and $\Gamma$ is of valency $m$;
(b) $v = 4$, $\Gamma[B, C] = K_{2,2}$, and $\Gamma$ is a connected graph of valency $4m$;
(c) $v = 4$, $\Gamma[B, C] = 2 \cdot K_2$, and $\Gamma$ is of valency $2m$. 
A map is a 2-cell embedding of a connected graph on a closed surface.
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A map $M$ is regular if its automorphism group $\text{Aut}(M)$ is regular on incident vertex-edge-face triples.
If $v = 3$ and $m = 1$, then $\Gamma$ can be obtained from a regular map of valency 3 by truncation.
Let $M$ be a 4-valent regular map, and let $G = \text{Aut}(M)$. 
Incident vertex-face pairs

- Let $M$ be a 4-valent regular map, and let $G = \text{Aut}(M)$.
- For each edge $\{\sigma, \sigma'\}$ of $M$, let $f, f'$ denote the faces of $M$ such that $\{\sigma, \sigma'\}$ is on the boundary of both $f$ and $f'$. 
Let $M$ be a 4-valent regular map, and let $G = \text{Aut}(M)$.

For each edge $\{\sigma, \sigma'\}$ of $M$, let $f$, $f'$ denote the faces of $M$ such that $\{\sigma, \sigma'\}$ is on the boundary of both $f$ and $f'$.

Let $O_\sigma(f)$ and $O_\sigma(f')$ be the other two faces of $M$ incident with $\sigma$ and opposite to $f$ and $f'$ respectively.
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Let $O_\sigma(f)$ and $O_\sigma(f')$ be the other two faces of $M$ incident with $\sigma$ and opposite to $f$ and $f'$ respectively.

Define $O_{\sigma'}(f)$ and $O_{\sigma'}(f')$ similarly.
Incident vertex-face pairs

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\[ O_\sigma(f) \quad O_\sigma'(f) \]

\[ f \quad f' \]

\[ \sigma \quad \sigma' \]
We construct four graphs $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, $\Gamma_4$ as follows.

They have vertices the incident vertex-face pairs of $M$ such that for each edge $\{\sigma, \sigma'\}$ of $M$:

- $\Gamma_1$: $(\sigma, f) \sim (\sigma', f), (\sigma, f') \sim (\sigma', f')$
- $\Gamma_2$: $(\sigma, f) \sim (\sigma', f'), (\sigma, f') \sim (\sigma', f)$
- $\Gamma_3$: $(\sigma, O_\sigma(f)) \sim (\sigma', O_{\sigma'}(f)), (\sigma, O_\sigma(f')) \sim (\sigma', O_{\sigma'}(f'))$
- $\Gamma_4$: $(\sigma, O_\sigma(f)) \sim (\sigma', O_{\sigma'}(f')), (\sigma, O_\sigma(f')) \sim (\sigma', O_{\sigma'}(f))$
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  - $\Gamma_3$: $(\sigma, O_\sigma(f)) \sim (\sigma', O_{\sigma'}(f))$, $(\sigma, O_\sigma(f')) \sim (\sigma', O_{\sigma'}(f'))$
  - $\Gamma_4$: $(\sigma, O_\sigma(f)) \sim (\sigma', O_{\sigma'}(f'))$, $(\sigma, O_\sigma(f')) \sim (\sigma', O_{\sigma'}(f))$

These are examples of graphs in case (c) with $m = 1$. 
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These are examples of graphs in case (c) with $m = 1$.

No example is known in case (c) with $m > 1$. 
Let $p$ be a prime such that $p \equiv 1 \pmod{16}$, and let $G = PSL(2, p)$. 
A group-theoretic construction

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$H = \langle a \rangle : \langle b \rangle \cong D_{16}, \langle a^4, b \rangle \cong \mathbb{Z}_2^2, \ N_G(\langle a^4, b \rangle) = S_4.$
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There exists an involution $g \in N_G(\langle a^4, b \rangle) \setminus \langle a^2, b \rangle$ such that $g$ interchanges $a^4$ and $b$. 
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There exists an involution $g \in \mathbf{N}_G(\langle a^4, b \rangle) \setminus \langle a^2, b \rangle$ such that $g$ interchanges $a^4$ and $b$.

Let $L = \langle a^4, ba \rangle \cong \mathbb{Z}_2^2$. 
Define $\Sigma := \text{Cos}(G, H, HgH)$, $\Gamma := \text{Cos}(G, L, LgL)$.

$\Gamma$ and $\Sigma$ are $G$-symmetric, connected and of valency 4.

Let $B := \left[H : L\right]$ and $B := \{B_x : x \in G\}$.

$B$ is a $G$-invariant partition of $V(\Gamma) = \left[G : L\right]$ such that $k = v - 2 = 2$, $\Gamma_B = C_4$, $\Gamma[B, C] = K_2^2$, and $\Sigma \sim \Gamma_B$.

These are examples of graphs in case (b) with $m = 1$.

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Define $\Sigma := \text{Cos}(G, H, HgH)$, $\Gamma := \text{Cos}(G, L, LgL)$.

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A group-theoretic construction

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  - $\Gamma$ and $\Sigma$ are $G$-symmetric, connected and of valency 4.

- Let $B := [H : L]$ and $\mathcal{B} := \{B^x : x \in G\}$.

  - $\mathcal{B}$ is a $G$-invariant partition of $\mathcal{V}(\Gamma) = \text{Cos}(G, L)$ such that $k = v - 2 = 2$, $\Gamma^{\mathcal{B}} = C_4$, $\Gamma[B, C] = K_{2,2}$ and $\Sigma \cong \Gamma_{\mathcal{B}}$. 

These are examples of graphs in case (b) with $m_1 = 1$.

No example is known in case (b) with $m_1 > 1$. 

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These are examples of graphs in case (b) with $m = 1$.

No example is known in case (b) with $m > 1$. 
Corollary

[Li, Praeger, Z] There exists an infinite family of connected symmetric graphs \( \Gamma \) of valency 4 which have a quotient graph \( \Gamma_B \) of valency 4 such that \( \Gamma \) is not a cover of \( \Gamma_B \).

This is the first (infinite) family of graphs with these properties.
In the case where \( k = \nu - 2 \geq 1 \) and \( \Gamma^B \) is connected, is \( \nu \) bounded by some function of the valency of \( \text{Simple}(\Gamma^B) \)?
Questions

- In the case where $k = v - 2 \geq 1$ and $\Gamma^B$ is connected, is $v$ bounded by some function of the valency of $\text{Simple}(\Gamma^B)$?

- Can $\Gamma$ be determined for small values of $m$?
In the case where \( k = v - 2 \geq 1 \) and \( \Gamma^B \) is connected, is \( v \) bounded by some function of the valency of \( \text{Simple}(\Gamma^B) \)?

Can \( \Gamma \) be determined for small values of \( m \)?

Study the case when \( \text{Simple}(\Gamma^B) = C_v \) and \( m \) is small (e.g. \( m = 3 \) in case (a), \( m = 2, 3 \) in case (b) and \( m = 2 \) in case (c)).