Solutions to Homework 3 - Math 2000

Notes: Check the back of the book for solutions to 2.1,2.5,2.9. 2.11 was done during tutorial.

(# 2.2) Consider the sets A, B, C, and D defined below. Which of the following statements are true? Give an explanation for each false statement.

$$\begin{aligned} A &= \{1, 4, 7, 10, 13, 16, \ldots\} \\ B &= \{x \in \mathbb{Z} \mid x \text{ is odd } \} \\ C &= \{x \in \mathbb{Z} \mid x \text{ is prime and } x \neq 2 \} \\ D &= \{1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}. \end{aligned}$$

Solution. (a). Note that A may also be written as

$$A = \{n \in \mathbb{N} \mid n = 3x + 1 \text{ with } x \in \mathbb{Z}\}.$$

However, $25 = 3 \cdot 8 + 1$ and thus $25 \in A$. Therefore $25 \in A$ is true.
(b). Since 33 is odd, it follows that $33 \in D$ is true.
(c). $22 \notin A \cup D$ is false. Note that $22 = 3 \cdot 7 + 1$ and thus $22 \in A$ and hence $22 \in A \cup D$ too.
(d). $C \subseteq B$ is false since $7 \in C$ but $7 \notin B$.
(e). $\phi \in B \cap D$ is false since ϕ is a subset but not an element of $B \cap D$.
(f) $53 \notin C$ is false since 53 is a prime number. (It suffices to check whether 53 is divisible by any primes $< \sqrt{53}$.)

(# 2.6) For the open sentence $P(A) : A \subseteq \{1, 2, 3\}$ over the domain $S = \mathcal{P}(\{1, 2, 4\})$, determine:
(a). all $A \in S$ for which $P(A)$ is true;
(b). all $A \in S$ for which $P(A)$ is false;
(c). all $A \in S$ for which $A \cap \{1, 2, 3\} = \phi$.
Solution. First note that

$$S = \mathcal{P}(\{1, 2, 4\}) = \{\phi, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}\}.$$

(a). Therefore $P(A)$ is true if A is one of:

$$\phi, \{1\}, \{2\}, \{1, 2\}.$$

(b). Likewise $P(A)$ is false if A is one of:

$$\{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}.$$

(c). Lastly $A \cap \{1, 2, 3\} = \phi$ if A is one of:

$$\phi, \{4\}.$$

(# 2.8) State the negation of each of the following statements:
(a). $\sqrt{2}$ is a rational number.
(b). 0 is not a negative integer.
(c). 111 is a prime number.
Solution.
(a) $\sqrt{2}$ is not a rational number.
(b) 0 is a negative integer.
(c) 111 is not a prime number.

(# 2.10) Let P: 15 is odd. and Q: 21 is prime. State each of the following in words, and determine whether they are true or false.
Solution. First note that P : 15 is odd. is true and Q: 21 is prime. is false since $21 = 3 \cdot 7$.
(a). $P \vee Q$ is true.
(b). $P \wedge Q$ is false.
(c). $\sim P$ is false and Q is false. Therefore $\sim P \vee Q$ is false.
(d). P is true and $\sim Q$ is true. Therefore $P \wedge \sim Q$ is true.

(# 2.14) Consider the statments P : 17 is even. and Q : 19 is prime. Write each of the following statements in words, and indicate whether it is true or false.
Solution. Note that P: 17 is even. is false and Q: 19 is prime. is true.
(a). $\sim P$ is the statement '17 is not even.' $\sim P$ is true.
(b). $P \vee Q$ is the statement '17 is even or 19 is prime.' $P \vee Q$ is true.
(c). $P \wedge Q$ is the statement '17 is even and 19 is prime.' $P \wedge Q$ is false.
(d). $P \implies Q$ is the statemnt '17 is even implies 19 is prime.' $P \implies Q$ is true.

(# 2.17) Consider the statements:
P: $\sqrt{2}$ is rational.
Q: $\frac{2}{3}$ is rational.
R: $\sqrt{3}$ is rational.
Solution. We begin by observing that P is false, Q is true, and R is false.
(a). Therefore $P \wedge Q$ is false and R is false. Thus $P \wedge Q \implies R$ is true.
(b). $P \wedge Q$ is false and $\sim R$ is true. Thus $P \wedge Q \implies \sim R$ is false.
(c). $\sim P$ is true and thus $\sim P \wedge Q$ is true. Since R is false it follows that $\sim P \wedge Q \implies R$ is false.
(d). $P \vee Q$ is true and $\sim R$ is true. Therefore $P \vee Q \implies \sim R$ is true.