Solutions to Homework 4 - Math 2000

(# 2.18) Consider the open sentences $P(n)$: $5n + 3$ is prime. and $Q(n)$: $7n + 1$ is prime over the domain $\mathbb{N}$.
(a). State $P(n) \implies Q(n)$ in words.
(b). State $P(2) \implies Q(2)$ in words. Is this statement true or false?
(c). State $P(6) \implies Q(6)$ in words. Is this statement true or false?
Solution. (a). If $5n + 1$ is prime, then $7n + 1$ is prime.
(b). If $5(2) + 1$ is prime, then $7(2) + 1$ is prime. Since $5(2) + 1 = 11$, $P(2)$ is true and $7(2) + 1 = 15$ thus $Q(2)$ is false. Therefore $P(2) \implies Q(2)$ is false.
(c). If $5(6) + 1$ is prime, then $7(6) + 1$ is prime. Since $5(6) + 1 = 31$, $P(5)$ is true and $7(6) + 1 = 43$ thus $Q(7)$ is true. Therefore $P(7) \implies Q(7)$ is true.

(# 2.20) In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. Determine all $x \in S$ for which $P(x) \implies Q(x)$ is a true statement.
(a). $P(x) : x - 3 = 4$; $Q(x) : x \geq 8$; $S = \mathbb{R}$.
(b). $P(x) : x^2 \geq 1$; $Q(x) : x \geq 1$; $S = \mathbb{R}$.
(c). $P(x) : x^2 \geq 1$; $Q(x) : x \geq 1$; $S = \mathbb{N}$.
(d). $P(x) : x \in [-1, 2]$; $Q(x) : x^2 \leq 2$; $S = [-1, 1]$.
Solution. (a). $P(x)$ is true if $x = 7$ and $P(x)$ is false if $x \neq 7$. $Q(x)$ is true if $x \geq 8$ and $Q(x)$ is false if $x < 8$. We know that $P(x) \implies Q(x)$ is false only if $P(x)$ is true and $Q(x)$ is false. This only occurs if $x = 7$. Therefore $P(x) \implies Q(x)$ is true for all other x. That is, it is true if $x \neq 7$.

(# 2.24) For the following open sentences $P(x)$ and $Q(x)$ over a domain S, determine all values of $x \in S$ for which the biconditional is true.
(a). $P(x) : |x| = 4$; $Q(x) : x = 4$; $S = \{-4, -3, 1, 4, 5\}$.
(b). $P(x) : x \geq 3$; $Q(x) : 4x - 1 > 12$; $S = \{0, 2, 3, 4, 6\}$.
(c). $P(x) : x^2 = 16$; $Q(x) : x^2 - 4x = 0$; $S = \{-6, -4, 0, 3, 4, 8\}$.
Solution. (a).

x	$P(x) : \lvert x\rvert = 4.$	$Q(x) : x = 4.$	$P(x) \Leftrightarrow Q(x)$
-4	T	F	F
-3	F	F	T
1	F	F	T
4	T	T	T
5	F	F	T

The biconditional is true for $x = -3, 1, 4, 5$.

(b).

x	$P(x) : x \geq 3.$	$Q(x) : 4x - 1 > 12.$	$P(x) \Leftrightarrow Q(x)$
0	F	F	T
2	F	F	T
3	T	F	F
4	T	T	T
6	T	T	T

Therefore the biconditional is true for $x = 0, 2, 4, 6$.

(c).

x	$P(x) : x^2 = 16.$	$Q(x) : x^2 - 4x = 0.$	$P(x) \Leftrightarrow Q(x)$
-6	F	F	T
-4	T	F	F
0	F	T	F
3	F	F	T
4	T	T	T
8	F	F	T

Therefore the biconditional is true for $x = -6, 3, 4, 8$.

(# 2.26) For the open sentences $P(x) : |x - 3| < 1$ and $Q(x) : x \in (2, 4)$. over the domain $\mathbb{R}$, state the biconditional in two different ways.
Solution. We have

' $|x - 3| < 1$ is equivalent to $x \in (2, 4)$.'

and

' $|x - 3| < 1$ if and only if $x \in (2, 4)$.'

and

' $|x - 3| < 1$ is a necessary and sufficient condition for $x \in (2, 4)$.'

(# 2.28) Let $S = \{1, 2, 3\}$. Consider the following open sentences over the domain S:

$$P(n) : \frac{(n+4)(n+5)}{2} \text{ is odd}$$
$$Q(n) : 2^{n-2} + 3^{n-2} + 6^{n-2} > (2.5)^{n-1}.$$

Determine three distinct elements $a, b, c \in S$ such that $P(a) \implies Q(a)$ is false, $Q(b) \implies P(b)$ is false, and $P(c) \Leftrightarrow Q(c)$ is true.
Solution. Note that

$$\begin{aligned} P(1) :& \frac{5 \cdot 6}{2} = 15 \text{ is odd. true} \\ P(2) :& \frac{6 \cdot 7}{2} = 21 \text{ is odd. true} \\ P(3) :& \frac{7 \cdot 8}{2} = 28 \text{ is odd. false} \end{aligned}$$

In addition,

$$\begin{aligned} Q(1) :& 2^{1-2} + 3^{1-2} + 6^{1-2} > 2.5^{1-1}. \\ & \frac{1}{2} + \frac{1}{3} + \frac{1}{6} > 1. \\ & 1 > 1. \text{ false} \\ Q(2) :& 2^{2-2} + 3^{2-2} + 6^{2-2} > 2.5^{2-1}. \\ & 1 + 1 + 1 > 2.5. \\ & 3 > 2.5. \text{ true} \\ Q(3) :& 2^{3-2} + 3^{3-2} + 6^{3-2} > 2.5^{3-1}. \\ & 2 + 3 + 6 > 2.5^2. \\ & 11 > \frac{25}{4}. \text{ true} \end{aligned}$$

Note if $a = 1$ then $P(1) \implies Q(1)$ is false. If $b = 3$ then $Q(3) \implies P(3)$ is false. If $c = 2$ then $P(2) \Leftrightarrow Q(2)$ is true.

(# 2.30) For the statements P and Q, show that $P \implies (P \vee Q)$ is a tautology.
Solution. Note that we have the following truth table:

P	Q	$P \vee Q$	$P \implies (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(# 2.32) For statements P and Q, show that $(P \wedge (P \implies Q)) \implies Q$ is a tautology. Then state $(P \wedge (P \implies Q)) \implies Q$ in words.
Solution. Note that we have the following truth table:

P	Q	$P \implies Q$	$P \wedge (P \implies Q)$	$P \wedge (P \implies Q) \implies Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

In words, the statement reads:

'If P and if P, then Q, then Q.'

(# 2.34) For statements P and Q, the implication $(\sim P) \implies (\sim Q)$ is called the inverse of the implication $P \implies Q$.
(a). Use the truth table to show that these statements are not logically equivalent.
(b). Find another implication that is logically equivalent to $\sim P \implies \sim Q$ and verify your answer.
Solution. (a),(b). Note that we have the following truth table:

P	Q	$P \implies Q$	$\sim P$	$\sim Q$	$(\sim P) \implies (\sim Q)$	$Q \implies P$
T	T	T	F	F	T	T
T	F	F	F	T	T	T
F	T	T	T	F	F	F
F	F	T	T	T	T	T

Observe that $Q \implies P$ is logically eqivalent to $\sim P \implies \sim Q$.

(# 2.36) For statements P, Q, and R, use a truth table to show that each of the following pairs of statements are logically eqivalent.
(a). $(P \wedge Q) \Leftrightarrow P$ and $P \implies Q$.
(b). $P \implies (Q \vee R)$ and $(\sim Q) \implies ((\sim P) \vee R)$.
Solution. (a),(b). Note that we have the following truth tables:

P	Q	$P \wedge Q$	$(P \wedge Q) \Longleftrightarrow P$	$P \implies Q$
T	T	T	T	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Therefore $(P \wedge Q) \Leftrightarrow P \equiv P \implies Q$

P	Q	R	$Q \vee R$	$P \implies (Q \vee R)$	$\sim Q$	$\sim P$	$\sim P \vee R$	$\sim Q \implies (\sim P \vee R)$
T	T	T	T	T	F	F	T	T
T	T	F	T	T	F	F	F	T
T	F	T	T	T	T	F	T	T
F	T	T	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
T	F	F	F	F	T	F	F	F
F	F	F	F	T	T	T	T	T

Therefore $P \implies (Q \vee R)$ and $(\sim Q) \implies ((\sim P) \vee R)$ are logically equivalent.

(# 2.38) For statements P, Q, and R show that $(P \vee Q) \implies R$ and $(P \implies R) \wedge (Q \implies R)$ are logically equivalent.
Solution.

P	Q	R	$P \vee Q$	$(P \vee Q) \implies R$	$P \implies R$	$Q \implies R$	$(P \implies R) \wedge (Q \implies R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	F	T	T	T	T
F	T	F	T	F	T	F	F
T	F	F	T	F	F	T	F
F	F	F	F	T	T	T	T

Therefore $(P \vee Q) \implies R$ and $(P \implies R) \wedge (Q \implies R)$ are logically equivalent.