Solutions to Homework 8 - Math 2000

All solutions except 4.22,4.24, and 4.36 may be found in the book.

(# 4.22) (a) Prove that if r is a real number such that $0 < r < 1$, then

$$\frac{1}{r(1-r)} \geq 4.$$

(b) If the real number r in part (a) is an integer, is the implication true in this case? Explain.
Solution. Note that

$$\begin{aligned}
&\frac{1}{r(1-r)} \geq 4 \text{ is true} \quad \text{(multiply both sides of the equation by } r(1-r))\\
&\text{only if } 1 \geq 4r(1-r) \text{ is true} \quad \text{(add } 4r^2 - 4r \text{ to both sides)}\\
&\text{only if } 4r^2 - 4r + 1 \geq 0 \text{ is true} \quad \text{(factor)}\\
&\text{only if } (2r-1)^2 \geq 0 \text{ is true.}
\end{aligned}$$

However, this last equation is clearly true for all $r \in \mathbb{R}$ since the square of any real number is always non-negative. Thus we could reverse all the steps to prove what we want. That is, if $r \in \mathbb{R}$, then it follows that $(2r-1)^2 \geq 0$. Now we have

$$\begin{aligned}
(2r-1)^2 &\geq 0 \text{ (expand out)}\\
4r^2 - 4r + 1 &\geq 0 \text{ (subtract } 4r^2 - 4r \text{ from both sides)}\\
1 \geq 4r - 4r^2 &= 4r(1-r) \text{ (divide by } r(1-r) \text{ as long as } r \neq 0, 1)\\
\frac{1}{r(1-r)} &\geq 4 \text{ (as long as } r \neq 0, 1).
\end{aligned}$$

We have proven that $\frac{1}{r(1-r)} \geq 4$ for all $r \in \mathbb{R}$ excluding 0 and 1. Thus it is certainly also true for all $0 < r < 1$.
(b) By the first part, this identity is true for all integers except 0 and 1. In these exceptional cases this identity is undefined.

(# 4.24) Prove that for every two real numbers x and y,

$$|x + y| \geq |x| - |y|$$

Solution. We begin by proving a small Lemma:
Lemma For any two real numbers a, b

$$|ab| = |a||b|. \tag{1}$$

Proof of Lemma. We shall prove this by cases:
(i) a, b both positive
(ii) a, b both negative
(iii) one of a and b is positive and the other is negative.
Case (i). Suppose $a, b > 0$. Then $|a| = a, |b| = b$ since $a > 0, b > 0$. Also, $ab > 0$ as both $a > 0, b > 0$. Therefore $|ab| = ab$. It follows that

$$|ab| = ab = |a||b|$$

which establishes equation (1).
Case (ii). Suppose both $a, b < 0$. Write $a = -c$ and $b = -d$ with $c > 0, d > 0$. Then

$|a| = |-c| = c$ and $|b| = |-d| = d$. Also $ab = (-c)(-d) = cd$ since $(-1)^2 = 1$. Note that $cd > 0$. Therefore $|ab| = |cd| = cd$. It follows that

$$|ab| = cd = |a||b|$$

which establishes equation (1).
Case (iii). Suppose $a > 0$ and $b < 0$. Let $b = -d$ with $d > 0$. Then $|a| = a$, $|b| = |-d| = d$. Also $ab = a(-d) = -(ad)$. Thus $|ab| = |-ad| = ad$ since $ad > 0$. It follows that

$$|ab| = ad = |a||b|$$

which establishes equation (1). This completes the proof of the lemma.

Now we are in a position to prove the question. Let $x, y \in \mathbb{R}$. Notice that

$$|x| = |(x+y) + (-y)| \leq |x+y| + |-y| \text{ by the triangle inequality .}$$

By the Lemma above $|-y| = |(-1)y| = |(-1)||y| = 1|y| = |y|$. Therefore this equation is

$$|x| \leq |x+y| + |y|.$$

Substracting $|y|$ off both sides yields

$$|x| - |y| \leq |x+y|$$

as asserted.

(# 4.36) Prove that

$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$

for every two sets A and B.
Solution. The key point of the exercise is to notice the following. Suppose $x \in \overline{A \cap B}$. Then $x \notin A \cap B$. That is

$$\begin{aligned} x \in \overline{A \cap B} &= x \notin A \cap B \\ &= \sim (x \in (A \cap B)) \\ &= \sim (x \in A \wedge x \in B) \\ &\equiv \sim (x \in A) \vee \sim (x \in B) \\ &\equiv (x \in \overline{A}) \vee (x \in \overline{B}) \\ &= x \in \overline{A} \cup \overline{B}. \end{aligned} \tag{2}$$

It follows from these logical equivalences that if $x \in \overline{A \cap B}$ then $x \in \overline{A} \cup \overline{B}$. This means that

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}.$$

On the other hand, if $x \in \overline{A} \cup \overline{B}$, then looking at the bottom of equation (2) and working backwards to the top we see that $x \in \overline{A \cap B}$ and hence

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}.$$

We conclude that

$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$