Solutions to Homework 9 - Math 2000

All solutions except 5.16 may be found in the book.

We first prove the following lemma:

Lemma Let $a \in \mathbb{Z}$. If $3 \mid a^2$ then $3 \mid a$.
Proof. (by contrapositive) Instead we shall prove

$$\forall a \in \mathbb{Z}, \text{ if } 3 \nmid a, \text{ then } 3 \nmid a^2.$$

Suppose $3 \nmid a$. Then $a = 3k + 1$ for some $k \in \mathbb{Z}$ or $a = 3k + 2$ for some $k \in \mathbb{Z}$. Note that if $a = 3k + 1$ then

$$a^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1.$$

Thus $3 \nmid a^2$. If $a = 3k + 2$ then

$$a^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$$

and it follows that $3 \nmid a^2$.

(# 4.5.16) Prove that $\sqrt{3}$ is irrational.
Solution. We give a proof by contradiction. Assume, to the contrary, that $\sqrt{3} = \frac{a}{b}$ where a and b have no common factors. Then we have $3 = \frac{a^2}{b^2}$ or

$$3b^2 = a^2.$$

Thus $3 \mid a^2$ and by the above lemma $3 \mid a$. Let

$$a = 3k$$

for some $k \in \mathbb{Z}$. Thus

$$3b^2 = (3k)^2 = 9k^2$$

and dividing by 3 we get

$$b^2 = 3k^2.$$

Note that $3 \mid b^2$ and by the above lemma $3 \mid b$. Thus $b = 3l$ for some $l \in \mathbb{Z}$. In conclusion

$$a = 3k \text{ and } b = 3l$$

and thus a and b have the common factor 3. This contradicts what we assumed at the beginning and completes the proof.