SOLUTIONS QUIZ 6 - MATH 2000

Question 1
Let $x \in \mathbb{Z}$. If $3 \nmid x$, then $3 \mid x^2 - 1$.

Proof. We give a direct proof. Since $3 \nmid x$ it follows that either $x = 3k + 1$ for some $k \in \mathbb{Z}$ or $x = 3k + 2$ for some $k \in \mathbb{Z}$.

Case 1. $x = 3k + 1$ for some $k \in \mathbb{Z}$.
Then

$$x^2 - 1 = (3k + 1)^2 - 1 = 9k^2 + 6k + 1 - 1 = 3(3k^2 + 2k)$$

and $3 \mid x^2 - 1$ since $3k^2 + 2k \in \mathbb{Z}$.

Case 2. $x = 3k + 2$ for some $k \in \mathbb{Z}$.
Then

$$x^2 - 1 = (3k + 2)^2 - 1 = 9k^2 + 12k + 4 - 1 = 3(3k^2 + 4k + 1)$$

and $3 \mid x^2 - 1$ since $3k^2 + 4k + 1 \in \mathbb{Z}$.
Therefore in either case we have $3 \mid x^2 - 1$ and this completes the proof. $\square$

Question 2
Let $a, b \in \mathbb{Z}$. Show that if $a \equiv 5 (\text{mod } 6)$ and $b \equiv 3 (\text{mod } 4)$, then

$$4a + 6b \equiv 6 (\text{mod } 8).$$

Proof. Since $a \equiv 5 (\text{mod } 6)$ we have

$$a = 5 + 6k$$

with $k \in \mathbb{Z}$. Since $b \equiv 3 (\text{mod } 4)$

$$b = 3 + 4l$$

with $l \in \mathbb{Z}$. Thus

$$\begin{aligned} 4a + 6b - 6 &= 4(5 + 6k) + 6(3 + 4l) - 6 \\ &= 20 + 24k + 18 + 24l - 6 \\ &= 32 + 24k + 24l \\ &= 8(4 + 3k + 3l). \end{aligned}$$

Since $4 + 3k + 3l \in \mathbb{Z}$ it follows that $8 \mid 4a + 6b - 8$ or

$$4a + 6b \equiv 6 (\text{mod } 8).$$

$\square$