LARGE GAPS BETWEEN THE ZEROS OF THE RIEMANN ZETA FUNCTION

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Abstract: We show that the generalized Riemann hypothesis implies that there are infinitely many consecutive zeros of the zeta function whose spacing is three times larger than the average spacing. This is deduced from the calculation of the second moment of the Riemann zeta function multiplied by a Dirichlet polynomial averaged over the zeros of the zeta function.

1. Introduction

If the Riemann hypothesis (RH) is true then the non-trivial zeros of the Riemann zeta function, $\zeta(s)$, with positive imaginary part, may be written as $1/2 + i\gamma_n$ with $\gamma_n \in \mathbb{R}$ and $0 < \gamma_1 < \gamma_2 \leq \ldots$. Riemann noted that the argument principle implies that number of zeros of $\zeta(s)$ in the box with vertices $0, 1, 1 + iT$, and iT is $N(T) \sim (T/2\pi) \log(T/2\pi e)$. This implies that on average $(\gamma_{n+1} - \gamma_n) \approx 2\pi/\log \gamma_n$ and hence the average spacing of the sequence $\hat{\gamma}_n = \gamma_n \log \gamma_n / 2\pi$ is one. Montgomery [9] investigated the pair correlation of these numbers and he proposed the fundamental conjecture

$$(1.1) \quad \frac{1}{N} \#\{1 \leq j \neq k \leq N \mid a \leq \hat{\gamma}_j - \hat{\gamma}_k \leq b\} \sim \int_a^b \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx$$

for $0 < a < b$ as $N \to \infty$. Moreover, it is expected that the consecutive spacings, $\hat{\gamma}_{n+1} - \hat{\gamma}_n$, have a limiting distribution function which matches the distribution of consecutive spacings of the eigenvalues of a large random Hermitian matrix. See Odlyzko [14] for extensive numerical evidence in favour of this conjecture and also see Rudnick-Sarnak [15] for a study of the n-level correlations of $\hat{\gamma}_n$. In light of the expected distribution of the consecutive spacings of zeta Montgomery suggested in [9] that there exist arbitrarily large and small gaps between the zeros of the zeta function. That is to say

$$\lambda = \limsup_{n \to \infty} (\hat{\gamma}_{n+1} - \hat{\gamma}_n) = \infty \text{ and } \mu = \liminf_{n \to \infty} (\hat{\gamma}_{n+1} - \hat{\gamma}_n) = 0 .$$

In this article, we focus on the large gaps and we assume the generalized Riemann hypothesis (GRH) is true. This conjecture states that the non-trivial zeros of the Dirichlet L-functions are on the $\mathrm{Re}(s) = 1/2$ line. We establish

Theorem 1. *The generalized Riemann hypothesis implies $\lambda > 3$.*

Selberg remarked in [16] that he could prove $\lambda > 1$. Montgomery and Odlyzko [10] obtained $\lambda > 1.9799$ assuming the Riemann hypothesis. This result was then improved by Conrey, Ghosh, and Gonek [2], [3] who obtained $\lambda > 2.337$ assuming RH and $\lambda > 2.68$ assuming GRH. The current record due to Hall is $\lambda > 2.34$. Remarkably, Hall's unconditional result is even better than what was previously known assuming RH. Hall's work makes use of Wirtinger's inequality in conjunction with asymptotic formulae for continuous mixed moments of the zeta function and its derivatives. Moreover, Hall is currently attempting to show that the asymptotic evaluation of all mixed moments of zeta and its derivatives yields $\lambda = \infty$. Theorem 1 extends the earlier work of Conrey et al. in [3]. Their work is based on the following idea of J. Mueller[11].

Let $H : \mathbb{C} \to \mathbb{R}_{\geq 0}$ be continuous and consider the associated functions

$$(1.2) \qquad \mathcal{M}_1(H,T) = \int_1^T H(1/2 + \mathrm{i}t)\, dt \ ,$$

$$(1.3) \qquad m(H,T;\alpha) = \sum_{T < \gamma < 2T} H(1/2 + \mathrm{i}(\gamma + \alpha)) \ ,$$

$$(1.4) \qquad \mathcal{M}_2(H,T;c) = \int_{-c/L}^{c/L} m(H,T;\alpha)\, d\alpha$$

where we put $L = \log(T/2\pi)$. This notation shall be used throughout the article. Note that

$$(1.5) \qquad \frac{\mathcal{M}_2(H,2T;c) - \mathcal{M}_2(H,T;c)}{\mathcal{M}_1(H,2T) - \mathcal{M}_1(H,T)} < 1$$

implies $\lambda > \frac{c}{\pi}$.

Mueller applied this idea with $H(s) = |\zeta(s)|^2$ and obtained $\lambda > 1.9$. Now consider the Dirichlet polynomial

$$(1.6) \qquad A(s) = \sum_{n \leq y} a(n) n^{-s} \ .$$

Assuming the Riemann hypothesis, Conrey et al. in [2] applied (1.5) to $H(s) = |A(s)|^2$ with $a(n) = d_{2.2}(n)$, $y = T^{1-\epsilon}$ and obtained $\lambda > 2.337$ (and $\mu < 0.5172$). Here $d_r(n)$ is the coefficient of n^{-s} in the Dirichlet series $\zeta(s)^r$. If r is a natural number then $d_r(n)$ equals the number of representations of n as a product of r positive integers. In recent work [12], we have shown that the Riemann hypothesis implies $\lambda > 2.56$ (and $\mu < 0.5162$). In [3], Conrey et al. applied (1.5) to $H(s) = |\zeta(s)A(s)|^2$ with $a(n) = 1$ and $y = (T/2\pi)^{\frac{1}{2}-\epsilon}$ and obtained $\lambda > 2.68$. However, in this situation it is necessary to assume GRH in order to evaluate the discrete mean value $m(H,T;\alpha)$.

We continue this programme by considering a more general choice for the coefficient $a(n)$. Precisely, we choose as our function $H_r(s) = |\zeta(s)A(s)|^2$ where $A(s)$ has coefficients

$$(1.7) \qquad a(n) = d_r(n) P\left(\frac{\log n}{\log y}\right)$$

for P a polynomial and for $r \in \mathbb{N}$. We are able to evaluate the desired quantities in (1.5) when $y = (T/2\pi)^\eta$ with $\eta < \frac{1}{2}$. Our work builds upon the work in [3], although there are significant complications arising from the fact that $d_r(n)$ is not completely multiplicative. Ideally, we would like to evaluate $m(|\zeta(s)A(s)|^2, T; \alpha)$ for an arbitrary Dirichlet polynomial A of length $y = (T/2\pi)^\eta$. In a recent preprint [13], we succeeded in evaluating

$$\sum_{0 < \gamma < T} \zeta'(\rho) A(\rho) A(1 - \rho)$$

for an arbitrary Dirichlet polynomial $A(s)$. Moreover, we did not assume that the coefficients of $A(s)$ are multiplicative. The method in [13] is slightly different as we instead follow the approach of [4]. The same method allows one to evaluate $m(|\zeta(s)A(s)|^2, T; \alpha)$ for an arbitrary Dirichlet polynomial.

If we take $H(s) = |\zeta(s)|^4$, then conjectures of Chris Hughes [8], based on random matrix theory, would allow us to evaluate (1.2)-(1.4). This led him (conjecturally) to $\lambda > 2.7$. In addition, taking $H(s) = |\zeta(s)|^k$ and assuming certain conjectures from RMT to evaluate (1.2)-(1.4), Hughes was able to obtain $\lambda > f(k)$ for some function $f(k)$ growing linearly to infinity as k goes to infinity. We may think of the choice $H_r(s) = |\zeta(s)A_r(s)|^2$ as a kind of approximation to $|\zeta(s)|^{2r+2}$.

We now state the precise result. We define several functions that will appear in the course of the proof. Given a polynomial P and $u \in \mathbb{Z}_{\geq 0}$ we define

$$(1.8) \qquad Q_u(x) = \int_0^1 \theta^u P(x + \theta(1-x)) \, d\theta .$$

Given $\vec{n} = (n_1, n_2, n_3, n_4, n_5) \in (\mathbb{Z}_{\geq 0})^5$ we define

$$(1.9) \quad i_P(\vec{n}) = \int_0^1 \int_0^{1-x} x^{r^2-1}(1-x)^{n_1}(1-y-x)^{n_2} y^{n_3} Q_{n_4}(x) Q_{n_5}(x+y) dy dx .$$

For $\eta \in \mathbb{R}$ and $\vec{n} = (n_1, n_2, n_3) \in (\mathbb{Z}_{\geq 0})^3$ we define

$$(1.10) \quad k_P(\vec{n}) = \int_0^1 \int_0^{1-x} x^{r-1}(\eta^{-1}-x)^{n_1} y^{r^2-1}(1-y)^{n_2} P(x+y) Q_{n_3}(y) dy dx .$$

Recall η corresponds to the length of our Dirichlet polynomial. Given $r \geq 1$ we define the constants
(1.11)

$$a_r = \prod_p \left((1 - p^{-1})^{r^2} \sum_{m=0}^\infty \left(\frac{\Gamma(r+m)}{\Gamma(r)m!} \right)^2 p^{-m} \right) , \quad C_r = \frac{a_{r+1}}{(r^2-1)!((r-1)!)^2} .$$

With all of these definitions in hand we present our result for $m(H_r, T; \alpha)$.

Theorem 2. *Suppose $r \in \mathbb{N}$ and $\eta < 1/2$. The generalized Riemann hypothesis implies*
(1.12)

$$m(H_r, T; \alpha) \sim \frac{C_r T L^{(r+1)^2+1}}{\pi} \mathrm{Re} \sum_{j=1}^\infty z^j \eta^{j+(r+1)^2+1} \left(\frac{r\hat{i}(r,j,\eta)}{j!} + \hat{k}(r,j,\eta) \right)$$

where $z = i\alpha L$, $|z| \ll 1$,
(1.13)
$$\hat{i}(r,j,\eta) = -i_P(r,r,j,r-1,r-1)\eta^{-1} + i_P(r+1,r,j,r,r-1) + i_P(r,r+1,j,r-1,r) ,$$
(1.14)

$$\hat{k}(r,j,\eta) = (r-1)! \sum_{n=-2}^{\min(j,r-2)} \frac{(-1)^{n+1} \binom{r}{n+2} k_P(j-n, r+n+2, r+n+1)}{(j-n)!(r+n+1)!} .$$

This result is valid up to an error term $O_{\epsilon,r}(TL^{(r+1)^2} + T^{1/2+\eta+\epsilon})$.

We note that it is possible to prove Theorem 2 only assuming the Generalized Lindelöf Hypothesis by following the work of Conrey, Ghosh, and Gonek [4] on simple zeros of $\zeta(s)$ and the recent preprint [13]. Even this

assumption may possibly be weakened further since the main theorem in [4] actually assumes an upper bound for the sixth integral moment of $L(s, \chi)$ on average.

As a check on our calculations we took $r = 1$ and $P(x) = 1$. After some calculation, Theorem 2 here reduces to

(1.15)

$$m(H_1, T; \alpha) \sim \frac{6}{\pi^2} \frac{TL^5}{2\pi} \sum_{j=0}^{\infty} \frac{(-1)^{j+1}(\alpha L)^{2j+2}}{(2j+5)!} \cdot$$

$$\cdot \left(\frac{-3\eta^2 + (2j+5)\eta^3}{3} - \frac{2j+5}{j+3} \eta^{2j+6} + \eta^{2j+7} + \eta^2(1-\eta)^{2j+5} \right) ,$$

in agreement with Theorem 1 of [3].

2. THEOREM 2 IMPLIES THEOREM 1

In this section, we deduce Theorem 1 from Theorem 2. The rest of the article will be devoted to establishing the discrete moment result of Theorem 2. Put $\eta = 1/2 - \epsilon$ with ϵ arbitrarily small. Since $\mathrm{Re}(z^j) = (-1)^k(\alpha L)^{2k}$ if $j = 2k$ and zero otherwise, it follows from (1.12) that

$$(2.1) \quad m(H_r, 2T; \alpha) - m(H_r, T; \alpha) = \phi(r, \eta, \alpha) \frac{C_r T L^{(r+1)^2+1}}{\pi}(1 + O(L^{-1}))$$

where

$$\phi(r, \eta, \alpha) = \eta^{(r+1)^2+1} \sum_{j=1}^{\infty} (-1)^j (\alpha L \eta)^{2j} \left(\frac{r\hat{i}(r, 2j, \eta)}{(2j+1)!} + \frac{\hat{k}(r, 2j, \eta)}{2j+1} \right) .$$

Integrating (2.1) with respect to α over the interval $[-c/L, c/L]$ we have $\mathcal{M}_2(H_r, 2T; c) - \mathcal{M}_2(H_r, T; c)$ equals

$$\frac{2C_r T L^{(r+1)^2} \eta^{(r+1)^2+1}}{\pi} \sum_{j=1}^{\infty} (-1)^j c^{2j+1} \eta^{2j} \left(\frac{r\hat{i}(r, 2j, \eta)}{(2j+1)!} + \frac{\hat{k}(r, 2j, \eta)}{2j+1} \right)$$

plus an error $O(TL^{(r+1)^2})$. In the above expression, we may replace $\eta = 1/2 - \epsilon$ by $1/2$ yielding

$$\mathcal{M}_2(H_r, 2T; c) - \mathcal{M}_2(H_r, T; c) = \frac{2C_r T L^{(r+1)^2} \eta^{(r+1)^2+1}}{\pi} \cdot$$

$$\sum_{j=1}^{\infty} \frac{(-1)^j c^{2j+1}}{2^{2j}} \left(\frac{r\hat{i}(r, 2j, \frac{1}{2})}{(2j+1)!} + \frac{\hat{k}(r, 2j, \frac{1}{2})}{(2j+1)} \right) + O(\epsilon T L^{(r+1)^2}) .$$

We now recall the following result of Conrey and Ghosh [1].

Lemma 2.1. If $y = T^\eta$ with $0 < \eta < 1/2$ then

$$\mathcal{M}_1(H_r, T) \sim \frac{a_{r+1}}{((r-1)!)^2(r^2-1)!} T(\log y)^{(r+1)^2}.$$

(2.2)

$$\int_0^1 \alpha^{r^2-1}(\eta^{-1}(1-\alpha)^{2r} Q_{r-1}(\alpha)^2 - 2(1-\alpha)^{2r+1} Q_r(\alpha) Q_{r-1}(\alpha)) \, d\alpha$$

as $T \to \infty$. *This is valid up to an error term which is $O(L^{-1})$ smaller than the main term.*

Hence, we have

$$\mathcal{M}_1(H_r, 2T) - \mathcal{M}_1(H_r, T) = C_r T (L\eta)^{(r+1)^2} \int_0^1 \alpha^{r^2-1}$$

$$\cdot (\eta^{-1}(1-\alpha)^{2r} Q_{r-1}(\alpha)^2 - 2(1-\alpha)^{2r+1} Q_{r-1}(\alpha) Q_r(\alpha)) \, d\alpha + O(\epsilon T L^{(r+1)^2}) \ .$$

We deduce that

$$\frac{\mathcal{M}_2(H_r, 2T; c) - \mathcal{M}_2(H_r, T; c)}{\mathcal{M}_1(H_r, 2T) - \mathcal{M}_1(H_r, T)} = f_r(c) + O(\epsilon)$$

where

$$(2.3) \qquad f_r(c) = \frac{1}{D} \sum_{j=1}^{\infty} \frac{(-1)^j c^{2j+1}}{2^{2j}} \left(\frac{r\hat{i}(r, 2j, \frac{1}{2})}{(2j+1)!} + \frac{\hat{k}(r, 2j, \frac{1}{2})}{2j+1} \right)$$

and

$$D := \pi \int_0^1 \alpha^{r^2-1} (\eta^{-1}(1-\alpha)^{2r} Q_{r-1}(\alpha)^2 - 2(1-\alpha)^{2r+1} Q_{r-1}(\alpha) Q_r(\alpha)) \, d\alpha \ .$$

We define $\lambda_r := \sup_{f_r(c)<1}(c)$ and thus $\lambda \geq \frac{\lambda_r}{\pi}$. We may now compute (2.3) for various choices of r and $P(x)$. For example, we shall choose $c = 3\pi$, $r = 2$ and $P(x) = x^{30}$. We compute the sum as follows: by a Maple calculation we have

$$D^{-1} \sum_{j=0}^{J} \frac{(-1)^j c^{2j+1}}{2^{2j}} \left(\frac{2\hat{i}(2, 2j, \frac{1}{2})}{(2j+1)!} + \frac{\hat{k}(2, 2j, \frac{1}{2})}{2j+1} \right) = 0.999481353\ldots$$

for $J = 30$. On the other hand, we may bound the terms $j > J$. Since $|Q_u(x)| \leq ||P||_1$ we have the crude bound

$$|i_P(\vec{n})| \leq \frac{||P||_1^2 (r^2-1)!(n_1+n_3+1)!}{(n_1+n_3+r^2+1)!(n_3+1)}$$

for $\vec{n} \in (\mathbb{Z}_{\geq 0})^5$. It thus follows that

$$|\hat{i}(r, 2j, 1/2)| \leq \frac{||P||_1^2 (r^2-1)!}{2j+1} \left(\frac{4(r+2j+1)!}{(r^2+r+2j+1)!} \right)$$

and hence

$$\left| \frac{1}{D} \sum_{j>J}^{\infty} \frac{(-1)^j c^{2j+1}}{2^{2j}} \frac{2\hat{i}(2, 2j, \frac{1}{2})}{(2j+1)!} \right| \leq \frac{48c||P||_1^2}{D(2J)} \sum_{j>J} \frac{(c/2)^{2j}(2j+3)!}{(2j+1)!(2j+7)!}$$

$$\leq \frac{48c||P||_1^2}{\sqrt{2\pi} D(2J)^5} \sum_{j>J} e^{-2j(\log(2j)-(\log(c/2)+1))}$$

$$< \frac{48c||P||_1^2}{\sqrt{2\pi} D(2J)^5} \frac{e^{-2J(\log(2J)-\log(c/2)-1)}}{2(\log(2J)-\log(c/2)-1)} < 10^{-45}$$

where we have applied $n! > (n/e)^n$. A similar calculation establishes that

$$\left| \frac{1}{D} \sum_{j>J}^{\infty} \frac{(-1)^j c^{2j+1}}{2^{2j}} \frac{\hat{k}(2, 2j, \frac{1}{2})}{(2j+1)} \right| < 10^{-22} \ .$$

We conclude that $f_2(3\pi) < 1$ and hence establish Theorem 1. We made our choice of r and $P(x)$ by a computer search. We note that there are many choices of r and $P(x)$ that improve the work of [3]. For example, $r = 3$, $P(x) = 1$ yields $\lambda > 2.78$ whereas $r = 2$, $P(x) = 1$ yields $\lambda > 2.86$. However, we have not made any serious attempt to find the optimal value given by this method as our primary goal was to exhibit other sequences that improved the work of [3]. It would be of some interest to find the optimal value of c that this method gives and it is likely that a more clever choice of P will improve Theorem 1.

3. Some notation and definitions

Throughout this article we shall employ the notation

$$(3.1) \qquad\qquad\qquad [t]_y := \frac{\log t}{\log y}$$

for $t, y > 0$. This will allow us to write several equations more compactly. In addition, we shall encounter a variety of arithmetic functions. We define $j_\tau(n)$, $\Lambda(n)$, and $d_r(n)$ as follows:

$$(3.2) \qquad\qquad\qquad j_\tau(n) = \prod_{p|n} (1 + O(p^{-\tau}))$$

for $\tau > 0$ and the constant in the O is fixed and independent of τ. Next $\Lambda(n)$ and $d_r(n)$ may be defined by their Dirichlet series generating functions:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \quad \text{and} \quad \zeta(s)^r = \sum_{n=1}^{\infty} \frac{d_r(n)}{n^s} \ .$$

Since this article concerns the calculation of discrete mean values of $m(H_r, T, \alpha)$ we need to invoke several properties of d_r. Throughout this article we apply repeatedly the following facts concerning d_r:

$$\sum_{a=0}^{\infty} d_r(p^a) p^{-as} = \left(1 - p^{-s}\right)^{-r} \ ,$$

$$(3.3) \qquad\qquad \sum_{m \le x} d_r(m) m^{-1} \ll \log^r x \ ,$$

$$\sum_{m \le x} d_r(m)^2 m^{-1} \ll \log^{r^2} x \ .$$

4. Initial manipulations

In this section we set up the plan of attack for our evaluation of $m(H_r, T; \alpha)$. Recall that T is large, $L = \log(T/2\pi)$, and ϵ can be made arbitrarily small. Let R denote the positively oriented contour with vertices $a+i, a+i(T+\alpha)$, $1-a+i(T+\alpha)$, $1-a+i$, the top edge of which has a small semicircular indentation centred at $1/2 + i(T+\alpha)$ opening downward and $a = 1 + O(L^{-1})$. By an application of Cauchy's residue theorem, the reflection principle, and RH we have

$$m(H_r, T; \alpha) = \frac{1}{2\pi i} \int_R \frac{\zeta'}{\zeta}(s - i\alpha)\zeta(s)\zeta(1-s)A(s)A(1-s)\, ds \ .$$

For s in the interior or boundary of R we have $A(s) \ll_\epsilon y^{1-\sigma+\epsilon}$ and $\zeta(s) \ll_\epsilon T^{(1-\sigma)/2+\epsilon}$. The first bound is elementary and the second is the convexity bound. These combine to give $\zeta(s)\zeta(1-s)A(s)A(1-s) \ll yT^{1/2+\epsilon}$. Now choose T' such that $T-2 < T' < T-1$ such that $T' + \alpha$ is not the ordinate of a zero of $\zeta(s)$ and $(\zeta'/\zeta)(\sigma + iT') \ll L^2$, uniformly for $-1 \le \sigma \le 2$. A simple argument using Cauchy's residue theorem establishes that the top edge of the contour is $yT^{1/2+\epsilon}$. Similarly, the bottom edge of the contour is $\ll_\epsilon yT^\epsilon$ since $|\zeta(s)| \ll 1$ for $|s| \ll 1$ and $|s - 1| \gg 1$. Differentiating the functional equation, $\zeta(1-s) = \chi(1-s)\zeta(s)$, we have

$$(4.1) \qquad \frac{\zeta'}{\zeta}(1 - s - i\alpha) = \frac{\chi'}{\chi}(1 - s - i\alpha) - \frac{\zeta'}{\zeta}(s + i\alpha) \ .$$

where $\chi(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)$. Now the right edge is

$$(4.2) \qquad I = \frac{1}{2\pi i} \int_{a+i}^{a+i(T+\alpha)} \frac{\zeta'}{\zeta}(s - i\alpha)\zeta(s)\zeta(1-s)A(s)A(1-s)\, ds$$

and the left edge is by (4.1)

$$\frac{1}{2\pi i} \int_{1-a+i(T+\alpha)}^{1-a+i} \frac{\zeta'}{\zeta}(s - i\alpha)\zeta(s)\zeta(1-s)A(s)A(1-s)\, ds =$$

$$\frac{1}{2\pi i} \int_{a-i(T+\alpha)}^{a-i} \left(\frac{\zeta'}{\zeta}(s + i\alpha) - \frac{\chi'}{\chi}(1 - s - i\alpha) \right) \zeta(s)\zeta(1-s)A(s)A(1-s)\, ds$$

$$= \overline{I} - \overline{J}$$

where

$$(4.3) \qquad J = \frac{1}{2\pi i} \int_{a+i}^{a+i(T+\alpha)} \frac{\chi'}{\chi}(1 - s + i\alpha)\zeta(s)\zeta(1-s)A(s)A(1-s)\, ds \ .$$

Combining our above calculations we obtain

$$(4.4) \qquad m(H_r, T; \alpha) = 2\mathrm{Re}I - \overline{J} + O_\epsilon(yT^{\frac{1}{2}+\epsilon}) \ .$$

We have reduced our calculation to the evaluation of I and J. The difficult term to evaluate is I. Thus we begin with the evaluation of J since it is rather simple. By Stirling's formula one has $(\chi'/\chi)(1 - s + i\alpha) =$

$-\log(t/2\pi) + O(t^{-1})$ for $t \geq 1$, $1/2 \leq \sigma \leq 2$, and $|\alpha| \leq cL^{-1}$. By moving the contour to the $1/2$ line in (4.3) and then substituting the previous estimate we obtain J equals

$$-\frac{1}{2\pi} \int_1^T (\log t/(2\pi))|\zeta A(1/2+it)|^2 \, dt + O\left(\int_1^T |\zeta A(1/2+it)|^2 \, \frac{dt}{t} + yT^{\frac{1}{2}+\epsilon}\right) .$$

The last term comes from the horizontal integral. An integration by parts shows that the second integral is $O(L^{(r+1)^2+1})$ and therefore

$$J = -\frac{L}{2\pi} \mathcal{M}_1(H_r, T) + \int_1^T \mathcal{M}_1(H_r, t) \frac{dt}{t} + O(L^{(r+1)^2+1} + yT^{\frac{1}{2}+\epsilon}))$$

where

$$\mathcal{M}_1(H_r, T) = \int_1^T H_r(1/2 + it) \, dt = \int_1^T |\zeta(1/2 + it)|^2 |A(1/2 + it)|^2 \, dt .$$

By Lemma 2.1 above, we deduce

$$
\begin{aligned}
J \sim -\frac{C_r T L^{(r+1)^2+1}}{2\pi} & \left(\eta^{(r+1)^2-1} \int_0^1 \alpha^{r^2-1}(1-\alpha)^{2r} Q_{r-1}(\alpha)^2 \, d\alpha \right. \\
(4.5) & \left. -2\eta^{(r+1)^2} \int_0^1 \alpha^{r^2-1}(1-\alpha)^{2r+1} Q_{r-1}(\alpha)Q_r(\alpha) \, d\alpha \right)
\end{aligned}
$$

which is valid up to an error term $O(L^{-1})$ smaller.

We have now reduced the evaluation of $m(H_r, T; \alpha)$ to that of I. We begin our evaluation of I with some initial simplifications. By the functional equation (4.2) becomes

$$I = \frac{1}{2\pi i} \int_{a+i}^{a+i(T+\alpha)} \chi(1-s)B(s)A(1-s) \, ds$$

where $B(s) = \frac{\zeta'}{\zeta}(s - i\alpha)\zeta^2(s)A(s) = \sum_{j=1}^{\infty} b(j)j^{-s}$ and

$$(4.6) \qquad b(j) = - \sum_{\substack{hmn=j \\ h \leq y}} d_r(h)P([h]_y) d(m)\Lambda(n)n^{i\alpha} .$$

However, Lemma 2 of [4] deals with such integrals.

Lemma 4.1. *Suppose* $B(s) = \sum_{j \geq 1} b(j)j^{-s}$ *and* $A(s) = \sum_{k \leq y} a(k)k^{-s}$ *where* $a(j) \ll d_{r_1}(j)(\log j)^{l_1}$ *and* $b(j) \ll d_{r_2}(j)(\log j)^{l_2}$ *for some non-negative integers* r_1, r_2, l_1, l_2 *and* $T^\epsilon \ll y \ll T$ *for some* $\epsilon > 0$. *If*

$$I = \int_{c+i}^{c+iT} \chi(1-s)B(s)A(1-s) \, ds$$

then

$$I = \sum_{k \leq y} \frac{a(k)}{k} \sum_{j \leq \frac{nT}{2\pi}} b(j)e(-j/k) + O(yT^{\frac{1}{2}}(\log T)^{r_1+r_2+l_1+l_2}) .$$

We deduce that

$$(4.7) \qquad I = \sum_{k \leq y} \frac{d_r(k)P([k]_y)}{k} \sum_{j \leq \frac{kT}{2\pi}} b(j)e\left(-j/k\right) + O(yT^{\frac{1}{2}+\epsilon}) \ .$$

The goal of the rest of this paper is to evaluate the sum in (4.7). We now give a brief sketch how the proof shall proceed. We define the Dirichlet series

$$\mathcal{Q}^*(s,\alpha,k) = \sum_{j=1}^{\infty} b(j)e(-j/k)j^{-s} \ .$$

The inner sum in (4.7) can be written by Perron's formula as

$$\frac{1}{2\pi i} \int_{(c)} \mathcal{Q}^*(s,\alpha,k) \left(\frac{kT}{2\pi}\right)^s \frac{ds}{s} = M(k) + E(k)$$

with $c > 1$. The last formula is obtained by moving this contour left to $\mathrm{Re}(s) = 1/2 + L^{-1}$. The main term, $M(k)$, arises from the residues of $\mathcal{Q}^*(s,\alpha,k)$ at $s = 1$ and $s = 1 + i\alpha$ and the error term, $E(k)$, is given by the integral along the line $\mathrm{Re}(s) = 1/2 + L^{-1}$. Thus

$$(4.8) \qquad I = \sum_{k \leq y} \frac{d_r(k)P([k]_y)M(k)}{k} + \sum_{k \leq y} \frac{d_r(k)P([k]_y)E(k)}{k} \ .$$

The first sum involving $M(k)$ will give the main term of our theorem and can be computed unconditionally. Nevertheless, the calculation is lengthy and complicated. The second sum with the $E(k)$ term requires the assumption of GRH. We now give a brief explanation of how the Generalized Riemann Hypothesis arises. Recall that $\mathcal{Q}^*(s,\alpha,k)$ is the Dirichlet series whose coefficient of j^{-s} is $b(j)e(-j/k)$. However, if $(j,k) = 1$ we may decompose $e(-j/k)$ into multiplicative characters as follows:

$$e(-j/k) = \frac{1}{\phi(k)} \sum_{\chi (\mathrm{mod}) \ k} \overline{\chi}(-j)\tau(\chi) \ .$$

It follows that $\mathcal{Q}^*(s,\alpha,k)$ may written as a linear combination of terms consisting of $L(s,\chi)$ and $(L'/L)(s,\chi)$. By assuming GRH we can show that $\mathcal{Q}^*(s,\alpha,k)$ only has poles at 1 and $1 + i\alpha$ which accounts for the main term $M(k)$. If GRH were false then $\mathcal{Q}^*(s,\alpha,k)$ would have extra poles arising from zeros that violate GRH. This obviously would complicate the argument. In dealing with the error term $E(k)$ we need a good bound for $\mathcal{Q}^*(s,\alpha,k)$. Since the Generalized Riemann Hypothesis implies the Generalized Lindelöf hypothesis, we may assume we have good bounds for $L(s,\chi)$ and $(L'/L)(s,\chi)$. As a consequence we obtain a good bound for $\mathcal{Q}^*(s,\alpha,k)$ and hence $E(k)$. It should be noted that the above argument is only valid for $(j,k) = 1$. If $(j,k) > 1$ there is a similar identity for $e(-j/k)$ and the same argument works.

5. Lemmas

In this section we present the lemmas that we require for bounding the contribution coming from the error terms, $E(k)$, and for evaluating the main term in (4.8). The next lemma is useful for analyzing Dirichlet series that are products of several other Dirichlet series.

Lemma 5.1. *Suppose that $A_j(s) = \sum_{n=1}^{\infty} \alpha_j(n) n^{-s}$ is absolutely convergent for $\sigma > 1$, for $1 \leq j \leq J$, and that*

$$A(s) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} = \prod_{j=1}^{J} A_j(s) \ .$$

Then for any positive integer d,

$$\sum_{n=1}^{\infty} \frac{\alpha(dn)}{n^s} = \sum_{d_1 \cdots d_J = d} \prod_{j=1}^{J} \left(\sum_{\substack{n=1 \\ (n, P_j) = 1}}^{\infty} \frac{\alpha_j(n d_j)}{n^s} \right)$$

where $P_j = \prod_{i<j} d_i$.

This is Lemma 3 of [4] pp.506.

In Lemmas 5.2 and 5.3 we consider the Dirichlet series $D(s, h/k)$ and $\mathcal{Q}(s, \alpha, h/k)$ which arise in the analysis of $\mathcal{Q}^*(s, \alpha, k)$.

Lemma 5.2. *For $(h, k) = 1$ with $k > 0$ we define*

$$D\left(s, h/k\right) = \sum_{n=1}^{\infty} d(n) n^{-s} e\left(nh/k\right) \ (\sigma > 1) \ .$$

Then $D(s, h/k)$ is regular in the entire complex plane except for a double pole at $s = 1$. Moreover, it has the same meromorphic part as $k^{1-2s} \zeta^2(s)$.

This is proven in Estermann [5] pp.124-126.

Lemma 5.3. *Let $(h, k) = 1$ and $k = \prod p^\lambda > 0$. For $\alpha \in \mathbb{R}$ and $\sigma > 1$ define*

$$(5.1) \qquad \mathcal{Q}\left(s, \alpha, h/k\right) = - \sum_{m,n=1}^{\infty} \frac{d(m) \Lambda(n)}{m^s n^{s-i\alpha}} e\left(\frac{-mnh}{k}\right) \ .$$

Then $\mathcal{Q}(s, \alpha, h/k)$ has a meromorphic continuation to the entire complex plane. If $\alpha \neq 0$, $\mathcal{Q}(s, \alpha, h/k)$ has
(i) at most a double pole at $s = 1$ with same principal part as

$$k^{1-2s} \zeta^2(s) \left(\frac{\zeta'}{\zeta}(s - i\alpha) - \mathcal{G}(s, \alpha, k) \right) \ ,$$

where

$$(5.2) \quad \mathcal{G}(s, \alpha, k) = \sum_{p|k} \log p \left(\sum_{a=1}^{\lambda-1} p^{a(s-1+i\alpha)} + \frac{p^{\lambda(s-1+i\alpha)}}{1 - p^{-s+i\alpha}} - \frac{1}{p^{s-i\alpha} - 1} \right) \ ;$$

(ii) a simple pole at $s = 1 + i\alpha$ with residue

$$-\frac{1}{k^{i\alpha}\phi(k)}\zeta^2(1 + i\alpha)\mathcal{R}_k(1 + i\alpha)$$

where

(5.3) $$\mathcal{R}_k(s) = \prod_{p^\lambda \| k} (1 - p^{-1} + \lambda(1 - p^{-s})(1 - p^{s-1})) \ .$$

Moreover, on GRH, $\mathcal{Q}(s, \alpha, h/k)$ is regular in $\sigma > 1/2$ except for these two poles.

This is Lemma 5 of [3] pp.217-218. The proof there proceeds by writing the generating function $\mathcal{Q}(s, \alpha, h/k)$ as a linear combination of $(L'/L)(s, \chi)$ where $L(s, \chi)$ is a Dirichlet L-function modulo k. These L-functions contribute the pole at $s = 1 + i\alpha$. Moreover, $\mathcal{Q}(s, \alpha, h/k)$ is regular for $\sigma > 1/2$ since $(L'/L)(s, \chi)$ is regular in this region assuming GRH.

For an arbitrary variable x we define the following generating function for d_r

(5.4) $$T_r(x, \lambda) = \sum_{j \geq \lambda} d_r(p^j)x^j \ .$$

Lemma 5.4. *For $r, \lambda \in \mathbb{N}$ and x an indeterminate we have*

$$(1 - x)^r T_r(x, \lambda) = \lambda d_r(p^\lambda) \int_0^x t^{\lambda-1}(1 - t)^{r-1} \, dt \ .$$

We define for $\lambda, r \in \mathbb{N}$ the polynomial

$$H_{\lambda,r}(x) := \lambda x^{-\lambda} \int_0^x t^{\lambda-1}(1 - t)^{r-1} \, dt \ .$$

Note that $H_{\lambda,r}(x)$ is a degree r polynomial and $H_{\lambda,r}(0) = 1$. Consequently, the lemma may be rewritten as

$$(1 - x)^r T_r(x, \lambda) = d_r(p^\lambda)x^\lambda H_{\lambda,r}(x) \ .$$

Proof. Define the generating functions

$$A(x, y) := \sum_{\lambda=1}^\infty (1 - x)^r T_r(x, \lambda)y^\lambda \ ,$$

$$B(x, y) := \sum_{\lambda=1}^\infty \left(\lambda d_r(p^\lambda) \int_0^x t^{\lambda-1}(1 - t)^{r-1} \, dt \right) y^\lambda \ .$$

We will show that these generating functions are equal and hence we establish the lemma. Note that

$$A(x, y) = (1 - x)^r \sum_{j=1}^\infty d_r(p^j)x^j \sum_{\lambda=1}^j y^\lambda = \frac{y(1 - x)^r}{y - 1} \sum_{j=1}^\infty d_r(p^j)x^j(y^j - 1)$$

$$= \frac{y}{y - 1} \left(\frac{(1 - x)^r}{(1 - xy)^r} - 1 \right)$$

and since $\lambda d_r(p^\lambda) = r d_{r+1}(p^{\lambda-1})$ for $\lambda \geq 1$

$$B(x,y) = r \int_0^x (1-t)^{r-1} \left(\sum_{\lambda=1}^\infty d_{r+1}(p^{\lambda-1}) t^{\lambda-1} y^\lambda \right) dt = ry \int_0^x \frac{(1-t)^{r-1}}{(1-ty)^{r+1}} \, dt \ .$$

A calculation shows that $A_x(x,y) = B_x(x,y) = \frac{ry(1-x)^{r-1}}{(1-xy)^{r+1}}$ and since $A(0,y) = B(0,y) = 0$ it follows that $A(x,y) = B(x,y)$.

Our calculations require Perron's formula.

Lemma 5.5. *Let $F(s) := \sum_{n \geq 1} a_n n^{-s}$ be a Dirichlet series with finite abscissa of absolute convergence σ_a. Suppose there exists a real number $\alpha \geq 0$ such that*

$$\sum_{n=1}^\infty |a_n| n^{-\sigma} \ll (\sigma - \sigma_a)^{-\alpha} \ (\sigma > \sigma_a)$$

and that B is a non-decreasing function such that $|a_n| \leq B(n)$ for $n \geq 1$. Then for $x, T \geq 2, \sigma \leq \sigma_a, \kappa := \sigma_a - \sigma + (\log x)^{-1}$, we have

(5.5)
$$\sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} F(s+w) \frac{x^w}{w} \, dw + O\left(\frac{x^{\sigma_a - \sigma} (\log x)^\alpha}{T} + \frac{B(2x)}{x^\sigma} \left(1 + x \frac{\log T}{T}\right) \right) \ .$$

This is Corollary 2.1 p.133 of [18].

The following Lemma is another place where GRH is invoked. This lemma gives bounds for $\mathcal{Q}^*(s, \alpha, k)$ in the critical strip. These bounds are required for estimating the error term $E(k)$ in (4.8). In fact, GRH shall be invoked in the form of a Lindelöf type bound for Dirichlet L-functions.

Lemma 5.6. *Assume GRH. Let $y = (T/2\pi)^\eta$ where $0 < \eta < 1/2$, $k \in \mathbb{N}$ with $k \leq y$, and $\alpha \in \mathbb{R}$. Set*

(5.6)
$$\mathcal{Q}^*(s, \alpha, k) = \sum_{j=1}^\infty b(j) j^{-s} e(-j/k) \ (\sigma > 1) \ ,$$

where

$$b(j) = - \sum_{\substack{hmn=j \\ h \leq y}} d_r(h) P([h]_y) d(m) \Lambda(n) n^{i\alpha} \ .$$

Then $\mathcal{Q}^*(s, \alpha, k)$ has an analytic continuation to $\sigma > 1/2$ except possible poles at $s = 1$ and $1 + i\alpha$. Furthermore,

$$\mathcal{Q}^*(s, \alpha, k) = O(y^{\frac{1}{2}} T^\epsilon)$$

where $s = \sigma + it$, $\frac{1}{2} + L^{-1} \leq \sigma \leq 1 + L^{-1}$, $|t| \leq T$, $|s - 1| > 0.1$, and $|s - 1 - i\alpha| > 0.1$.

Proof. If χ is a character mod k, its Gauss sum is $\tau(\chi) = \sum_{h=1}^{k} \chi(h)e\,(h/k)$ from which it follows that

$$e\,(-j/k) = \sum_{d|j, d|k} \frac{1}{\phi\,(k/d)} \sum_{\chi\,(\mathrm{mod}\,\frac{k}{d})} \tau(\overline{\chi})\chi(-j/d)\ .$$

By inserting this identity in (5.6) we obtain

$$\mathcal{Q}^*(s, \alpha, k) = \sum_{d|k} \frac{1}{\phi\,(k/d)\,d^s} \sum_{\chi\,(\mathrm{mod}\,\frac{k}{d})} \tau(\overline{\chi})\overline{\chi}\,(-d)\,B(s, d)$$

where for $\sigma > 1$, $B(s, d) = \sum_{j=1}^{\infty} b(jd)\chi(jd)j^{-s}$. We now write $P(x) = \sum_{i=0}^{N} c_i x^i$ and hence we obtain

$$(5.7) \qquad \mathcal{Q}^*(s, \alpha, k) = \sum_{i=0}^{N} \frac{c_i}{(\log y)^i} \mathcal{Q}_i^*(s, \alpha, k)$$

where

$$(5.8) \quad \mathcal{Q}_i^*(s, \alpha, k) = \sum_{d|k} \frac{1}{\phi\,(k/d)\,d^s} \sum_{\chi\,(\mathrm{mod}\,\frac{k}{d})} \tau(\overline{\chi})\overline{\chi}\,(-d)\,\frac{\partial^i}{\partial z^i}\,B(s, d; z)|_{z=0}\ ,$$

$$B(s, d; z) = \sum_{j=1}^{\infty} b_z(dj)\chi(dj)j^{-s}, \text{ and } b_z(j) = \sum_{\substack{hmn=j \\ h \leq y}} d_r(h)h^z d(m)\Lambda(n)n^{i\alpha}\ .$$

Since χ is completely multiplicative we note that

$$B(s, 1; z) = \left(\sum_{h \leq y} \frac{\chi(h)d_r(h)h^z}{h^s}\right) L(s, \chi)^2 \left(\sum_{n \geq 1} \frac{\chi(n)\Lambda(n)}{n^{s-i\alpha}}\right)\ .$$

An application of Lemma 5.1 implies
(5.9)
$$B(s, d; z) = \sum_{f_1 f_2 f_3 f_4 = d} \mathcal{A}_1(s, f_1; z)\mathcal{A}_2(s, f_2, f_1)\mathcal{A}_2(s, f_3, f_1 f_2)\mathcal{A}_3(s, f_4, f_1 f_2 f_3)$$

where

$$\mathcal{A}_1(s, f; z) = \chi(f) \sum_{h \leq y/f} \frac{\chi(h)d_r(fh)(fh)^z}{h^s}\ ,$$

$$(5.10) \qquad \mathcal{A}_2(s, f, r) = \sum_{(n,r)=1} \frac{\chi(fn)}{n^s} = \chi(f)L(s, \chi)\prod_{p|r}(1 - \chi(p)p^{-s})\ ,$$

$$\mathcal{A}_3(s, f, r) = -\sum_{(n,r)=1} \chi(fn)\Lambda(fn)(fn)^{i\alpha}n^{-s}\ .$$

We are aiming to show that uniformly for $|z| \leq 0.1L^{-1}$

$$(5.11) \qquad B(s, d; z) \ll_\epsilon \begin{cases} y^{\frac{1}{2}}T^\epsilon & \text{if } \chi \text{ is principal} \\ T^\epsilon & \text{otherwise} \end{cases}$$

in the region $\sigma \geq 1/2 + L^{-1}$, $|t| \leq T$, and $|s - 1|, |s - 1 - i\alpha| > 0.1$. If (5.11) holds then we have by applying the Cauchy integral formula with a circle of radius $0.1L^{-1}$ that

$$\frac{\partial^i}{\partial z^i} B(s, d; z)|_{z=0} \ll \left\{ \begin{array}{ll} y^{\frac{1}{2}} T^\epsilon & \text{if } \chi \text{ is principal} \\ T^\epsilon & \text{otherwise} \end{array} \right. .$$

By (5.8) and this last identity we have

$$\mathcal{Q}_i^*(s, \alpha, k) \ll_\epsilon T^\epsilon \sum_{d|k} \frac{1}{\phi(k/d)d^{\frac{1}{2}}} \left(y^{\frac{1}{2}} |\tau(\chi_0)| + \sum_{\chi \neq \chi_0 (\text{mod } k/d)} |\tau(\chi)| \right) .$$

Since $|\tau(\chi)| \ll 1$ if χ is principal and $|\tau(\chi)| \ll (k/d)^{1/2}$ otherwise

$$\mathcal{Q}_i^*(s, \alpha, k) \ll T^\epsilon \left((y/k)^{1/2} \sum_{d|k} d^{1/2} \phi(d)^{-1} + k^{1/2} \sum_{d|k} d^{-1} \right) \ll y^{1/2} T^\epsilon$$

and hence by (5.7) the desired bound $Q^*(s, \alpha, k) \ll yT^{1/2+\epsilon}$ follows. It now suffices to establish (5.11). If χ is principal (mod k/d) then

$$\mathcal{A}_1(s, f; z) \ll f^\epsilon \sum_{n \leq y/f} n^{-\frac{1}{2}} \ll y^{1/2} .$$

Now suppose χ is non-principal. If $y/f \ll y^\epsilon$, we have trivially that $|\mathcal{A}_1(s, f)| \ll y^\epsilon$. Otherwise $y/f \gg y^\epsilon$ and by Perron's formula

$$\mathcal{A}_1(s, f; z) = \frac{\chi(f)f^z}{2\pi i} \int_{\kappa-2iT}^{\kappa+2iT} G(s + z + w) \frac{(y/f)^w}{w} dw + O(1)$$

for $\sigma \geq 1/2 + L^{-1}$, $|t| \leq T$, $\kappa = 1 - \sigma + 2L^{-1}$ where $G(w) = \sum_{n=1}^\infty d_r(fn)\chi(n)n^{-w}$. By multiplicativity we have

$$G(w) = L(w, \chi)^r \prod_{p^e || f} \left(\frac{\sum_{a=0}^\infty \chi(p^a) d_r(p^{e+a}) p^{-aw}}{\sum_{a=0}^\infty \chi(p^a) d_r(p^a) p^{-aw}} \right)$$

and furthermore by Lemma 5.4 it follows that

$$G(w) = d_r(f) L(w, \chi)^r \prod_{p^\lambda || f} H_{\lambda,r}(x_p)$$

with $x_p = \chi(p)p^{-s}$. Since $|x_p| \leq p^{-\sigma}$ and $H_{\lambda,r}(0) = 1$ it follows that

$$\left| \prod_{p^\lambda || f} H_{\lambda,r}(x_p) \right| \ll \prod_{p || f} (1 + O(p^{-1/2})) \ll f^\epsilon .$$

In addition, GRH implies $|L(w, \chi)| \ll (1 + |t|)^\epsilon (k/d)^\epsilon$ for $\text{Re}(w) \geq 1/2$ and any $\epsilon > 0$. We now move the contour in the above integral to $\text{Re}(w) = \kappa'$ line where $\kappa' = 1/2 - \sigma + 2L^{-1}$ and we have

$$\mathcal{A}_1(s, f; z) = \frac{\chi(f)f^z}{2\pi i} \int_{\kappa'-2iT}^{\kappa'+2iT} G(s + z + w) \frac{(y/f)^w}{w} dw + O(T^\epsilon) .$$

Since $0.5 \leq \mathrm{Re}(s + z + w)$ and $\mathrm{Re}(w) \leq L^{-1}$ it follows that

$$\mathcal{A}_1(s, f; z) \ll f^\epsilon T^\epsilon (k/d)^\epsilon (y/f)^{L^{-1}} \int_{\kappa'-2iT}^{\kappa'+2iT} \frac{|dw|}{|w|} \ll T^\epsilon .$$

For f and r dividing d, we have $\mathcal{A}_j(s, f, r) \ll T^\epsilon$ for $j = 2, 3$. This is proven in [3] pp.219-220. By (5.9) in conjunction with our preceding bounds for the $\mathcal{A}_j$ we obtain (5.11) which finishes the lemma.

The purpose of the next five lemmas is to provide a variety of formulae for mean values of certain multiplicative functions which arise in our asymptotic evaluation of I (4.7). Lemma 5.7 provides bounds for certain divisor sums. Lemmas 5.8, 5.9, and 5.11 give asympotic formulae for divisor and other divisor-like sums. Lemma 5.10 provides a formula for simple prime number sums.

Lemma 5.7. *For $\alpha \in \mathbb{R}$ and $j \in \mathbb{Z}_{\geq 0}$ we have*

$$\mathcal{G}^{(j)}(1, \alpha, k) = \sum_{p|k} p^{i\alpha} (\log p)^{j+1} + O(C_j(k))$$

where $\mathcal{G}(s, \alpha, k)$ is defined by (5.2) and

$$C_j(k) = \sum_{p|k} \frac{\log^j p}{p} + \sum_{p^a||k,\ a \geq 2} a \log^j p .$$

Moreover, we have

$$\sum_{h,k \leq x} \frac{d_r(h) d_r(k)(h,k)}{hk} C_j\left(\frac{k}{(h,k)}\right) \ll (\log x)^{r^2+r} .$$

Proof. We remark that the first identity is proven in [3] pp.222-223. The sum we are considering is bounded by

$$\sum_{h,k \leq x} \frac{d_r(h) d_r(k)}{hk}(C_j(k) + 1) \sum_{\substack{a|h \\ a|k}} \phi(a)$$

$$\leq \sum_{a \leq x} \frac{d_r(a)^2 \phi(a)}{a^2} \sum_{h,k \leq \frac{x}{a}} \frac{d_r(h) d_r(k)(C_j(ak) + 1)}{hk}$$

$$\leq (\log x)^{2r} \sum_{a \leq x} \frac{d_r(a)^2(C_j(a) + 1)}{a} + (\log x)^r \sum_{a \leq x} \frac{d_r(a)^2}{a} \sum_{k \leq \frac{x}{a}} \frac{d_r(k) C_j(k)}{k} .$$

Observe that

$$\sum_{a \leq y} \frac{d_r(a)^2 C_j(a)}{a} = \sum_{p \leq y} \frac{\log^j p}{p} \sum_{u \leq \frac{y}{p}} \frac{d_r(up)^2}{up} + \sum_{p^a \leq y, a \geq 2} a(\log p)^j \sum_{u \leq \frac{y}{p^a}} \frac{d_r(up^a)^2}{up^a}$$

$$\ll (\log x)^{r^2} \left(\sum_p \frac{(\log p)^j}{p^2} \right) \ll (\log x)^{r^2}$$

where we have applied (3.3). A similar argument establishes that
$\sum_{k \leq x} d_r(k) C_j(k) k^{-1} \ll (\log x)^{r^2+r}$. Putting together the results establishes
the lemma.

We now introduce the arithmetic function $\sigma_r(m, s)$ where $r \in \mathbb{N}$ and
$s \in \mathbb{C}$. It is defined by

$$(5.12) \quad \sigma_r(m, s) := \left(\sum_{n=1}^{\infty} \frac{d_r(mn)}{n^s} \right) \zeta(s)^{-r} = \prod_{p^\lambda || m} (1 - p^{-s})^r p^{\lambda s} \sum_{j \geq \lambda}^{\infty} \frac{d_r(p^j)}{p^{js}} .$$

The second equation is obtained by mutiplicativity. By Lemma 5.4 it follows
that

$$\sigma_r(m, s) = \prod_{p^\lambda || m} d_r(p^\lambda) H_{\lambda, r}(p^{-s}) .$$

The value $s = 1$ will have a special importance so we set $\sigma_r(m) := \sigma_r(m, 1)$.
In the following calculations we shall often employ the bound

$$(5.13) \qquad |\sigma_r(m, s)| \ll d_r(m) j_\tau(m) \text{ for } \mathrm{Re}(s) \geq \tau > 0 .$$

The function σ_r is a correction factor that arises due to the fact d_r is not
completely mutiplicative. More precisely, we notice in all cases of the fol-
lowing lemma that

$$\sum_{h \leq t} d_r(mh) f(h) \sim \sigma_r(m) \sum_{h \leq t} d_r(h) f(h)$$

where f is a smooth function.

Lemma 5.8. *Suppose* $r, n \in \mathbb{N}$, $1 \leq x, n \leq \frac{T}{2\pi}$, *and* $F \in C^1([0, 1])$. *There
exists an absolute constant* $\tau_0 = \tau_0(r)$ *such that*

$$\sum_{h \leq x} \frac{d_r(nh)}{h} F([h]_x) = \frac{\sigma_r(n)(\log x)^r}{(r-1)!} \int_0^1 \theta^{r-1} F(\theta) \, d\theta + O(d_r(n) j_{\tau_0}(n) L^{r-1}) .$$

Suppose $m, u, v \in \mathbb{N}$, $1 \leq y, m \leq \frac{T}{2\pi}$, p *a prime with* $p \leq \frac{T}{2\pi}$, *and* $P \in
C^1([0, 1])$. *We now deduce the following formulae:*

$$(i) \sum_{h \leq \frac{y}{m}} \frac{d_r(mh)}{h} (\log h)^u P([mh]_y) \sim \frac{\sigma_r(m)}{(r-1)!} \log\left(\frac{y}{m}\right)^{r+u} \int_0^1 F_1(\theta, m) \, d\theta ,$$

$$(ii) \sum_{h \leq \frac{y}{mp}} \frac{d_r(mph) P([mph]_y)(\log ph)^v}{h} \sim \frac{\sigma_r(pm)}{(r-1)!} \log\left(\frac{y}{pm}\right)^r \int_0^1 F_2(\theta, pm) \, d\theta ,$$

$$(iii) \sum_{h \leq \frac{y}{m}} \frac{d_r(mh) P([mh]_y) \log(\frac{T}{2\pi h})^u}{h} \sim \frac{\sigma_r(m)(\log y)^{u+r}}{(r-1)!} \int_0^{1-[m]_y} F_3(\theta, m) \, d\theta ,$$

where these formulae are valid up to error terms $d_r(m)j_{\tau_0}(m)L^{r+u-1}$, $d_r(m)j_{\tau_0}(m)L^{r+v-1}$, $d_r(m)j_{\tau_0}(m)L^{r+u-1}$ *respectively and*

$$F_1(\theta, m) = \theta^{r+u-1} P\left([m]_y + (1 - [m]_y)\theta\right) \ ,$$

$$(5.14) \quad F_2(\theta, pm) = \theta^{r-1} \left(\log p + \theta \log \frac{y}{pm} \right)^v P\left([pm]_y + (1 - [pm]_y)\theta\right) \ ,$$

$$F_3(\theta, m) = \theta^{r-1}(\eta^{-1} - \theta)^u P\left([m]_y + \theta\right) \ .$$

Proof. By the methods of [17]

$$\sum_{h \leq t} \frac{d_r(nh)}{h} = \frac{\sigma_r(n)(\log t)^r}{r!} + O(d_r(n)j_{\tau_0}(n)L^{r-1})$$

for some $\tau_0 = \tau_0(r) > 0$. We abbreviate this last equation to $T(t) = M(t) + O(\epsilon(n))$. For $g \in \mathcal{C}^1([0, 1])$ we have

$$\sum_{h \leq x} \frac{d_r(nh)}{h} g\left([h]_x\right) = \int_1^x M'(t)g\left([t]_x\right) dt$$

$$+ O\left(\epsilon(n) \left(|g(0)| + |g(1)| + \frac{1}{\log x} \int_1^x |g'\left([t]_x\right)| \frac{dt}{t} \right) \right) \ .$$

The error term is $\ll \epsilon(n)$ and the principal term is

$$\frac{\sigma_r(n)}{(r-1)!} \int_1^x (\log t)^{r-1} g\left([t]_x\right) dt = \frac{\sigma_r(n)(\log x)^{r-1}}{(r-1)!} \int_0^1 \theta^{r-1} g(\theta) \, d\theta$$

by the variable change $\theta = [t]_x$. This yields the first formula. Formulae $(i) - (iii)$ correspond to the following choices of parameters $(n, g(\theta), x)$:

$$\left(m \ , \theta^u P([m]_y + \theta), \frac{y}{m}\right) \ , \left(pm \ , ([p]_x + \theta)^u \, P([pm]_y + \theta), \frac{y}{pm}\right) \ ,$$

$$\left(m \ , \left(\frac{\log \frac{T}{2\pi}}{\log x} - \theta\right)^u P([m]_y + \theta), \frac{y}{m}\right) \ .$$

We remark that equation (iii) requires the variable change $\theta \to [x]_y\theta$.

In the following lemma we consider averages of the expression $\sigma_r(\cdot)^2$. It is in this lemma that the constant a_{r+1} (1.11) of Theorem 2 appears. It naturally arises upon considering the Dirichlet series $\sum_{n \geq 1} \phi(n)\sigma_r(n)^2 n^{-s}$.

Lemma 5.9. *Let* $r \in \mathbb{N}$ *and* $g \in C^1([0, 1])$.
(i) For $p \leq y$ *prime we have*

$$\sum_{m \leq \frac{y}{p}} \frac{\phi(m)\sigma_r(m)\sigma_r(pm)}{m^2} g\left([m]_y\right) = \frac{\sigma_r(p)a_{r+1}(\log y)^{r^2}}{(r^2 - 1)!} \int_0^{1 - [p]_y} \delta^{r^2 - 1} g(\delta) \, d\delta$$

$$+ O\left((\log y)^{r^2} (p^{-1} + (\log y)^{-1}) \right) \ .$$

(ii) For $0 \leq \theta < 1$ *we have*

$$\sum_{m \leq y^{1-\theta}} \frac{\phi(m)\sigma_r(m)^2}{m^2} g\left([m]_y\right) = \frac{a_{r+1}(\log y)^{r^2}}{(r^2 - 1)!} \int_0^{1-\theta} \delta^{r^2 - 1} g(\delta) \, d\delta (1 + O((\log y)^{-1})) \ .$$

Proof. We only prove (i) since (ii) is similar. We begin by noting that

$$\sum_{m \leq t} \frac{\phi(m)\sigma_r(m)\sigma_r(pm)}{m^2} = \sigma_r(p) \sum_{m \leq t} \frac{\phi(m)\sigma_r(m)^2}{m^2}$$

$$+ \sum_{\substack{m \leq t \\ p|m}} \frac{\phi(m)\sigma_r(m)(\sigma_r(p)\sigma_r(m) - \sigma_r(pm))}{m^2} \ .$$

Since $\sigma_r(m) \ll d_r(m)j_1(m)$, $d_r(uv) \leq d_r(u)d_r(v)$, and $\phi(up) \leq \phi(u)p$, it follows that the second term is

$$\ll \frac{d_r(p)}{p} \sum_{n \leq \frac{x}{p}} \frac{d_r(n)^2 j_1(n)}{n} \ll p^{-1}(\log x)^{r^2} \ .$$

By equations (36)-(38) of [1] in conjunction with Theorem 2 of [17] we deduce

$$\sum_{m \leq t} \frac{\phi(m)\sigma_r(m)^2}{m^2} = \frac{a_{r+1}(\log t)^{r^2}}{(r^2 - 1)!}(1 + O((\log t)^{-1}))$$

and hence we arrive at

$$\sum_{m \leq t} \frac{\phi(m)\sigma_r(m)\sigma_r(pm)}{m^2} = \frac{ra_{r+1}(\log t)^{r^2}}{(r^2 - 1)!} + O\left((\log t)^{r^2}p^{-1} + \log^{r^2-1} t\right) \ .$$

We abbreviate this equation to $T(t) = M(t) + O(E(t))$. The sum in (i) may be expressed as the Stieltjes integral

$$\int_{1^-}^{\frac{y}{p}} g\left([t]_y\right) dT(t) = \int_{1^-}^{\frac{y}{p}} g\left([t]_y\right) dM(t) + g\left([t]_y\right) E(t)|_{1^-}^{\frac{y}{p}} - \int_{1^-}^{\frac{y}{p}} g'\left([t]_y\right) E(t) dt \ .$$

The integral equals

$$\frac{ra_{r+1}}{(r^2 - 1)!} \int_1^{\frac{y}{p}} (\log t)^{r^2-1} g\left([t]_y\right) \, dt = \frac{ra_{r+1}}{(r^2 - 1)!} \int_0^{1-[p]_y} \delta^{r^2-1} g(\delta) \, d\delta \ .$$

Moreover, it is clear that the corresponding error term is $O\left((\log t)^{r^2}p^{-1} + (\log t)^{r^2-1}\right)$.

In the main calculation of this article we compute certain simple sums over primes. The following lemma provides the required result

Lemma 5.10. *Suppose $w \geq 1$, $0 \leq \theta < 1$, and $g \in C^1([0,1])$ then*

$$\sum_{p \leq y^{1-\theta}} \frac{(\log p)^w}{p^{1-i\alpha}} g\left([p]_y\right) = \sum_{j=0}^{\infty} \frac{(i\alpha)^j}{j!} (\log y)^{j+w} \int_0^{1-\theta} \beta^{j+w-1} g(\beta) \, d\beta$$

$$+ O((\log y)^{w-1}) \ .$$

Proof. By Stieltjes integration the sum in question is

$$\int_1^{y^{1-\theta}} t^{i\alpha} (\log t)^{w-1} g\left([t]_y\right) \frac{d\theta(t)}{t}$$

where $\theta(t) = \sum_{p \leq t} \log p = t + \epsilon(t)$ and $\epsilon(t) \ll t \exp(-c\sqrt{\log t})$. Note that the main term is

$$\int_1^{y^{1-\theta}} t^{i\alpha}(\log t)^{w-1} g\left([t]_y\right) \frac{dt}{t} = \sum_{j=0}^{\infty} \frac{(i\alpha)^j}{j!} \int_1^{y^{1-\theta}} (\log t)^{j+w-1} g\left([t]_y\right) \frac{dt}{t} .$$

By the variable change $\beta = [t]_y$ we obtain the required expression for the principal part. Put $h(t) = t^{i\alpha}(\log t)^{w-1} g\left([t]_y\right) t^{-1}$ and note $h(t) \ll (\log t)^{w-1} t^{-1}$ and $h'(t) \ll (\log t)^{w-1} t^{-2}$ for $t \leq y$. By the above bound for $\epsilon(t)$

$$\int_1^{y^{1-\theta}} h(t) d\epsilon(t) \ll h(y^{1-\theta})\epsilon(y^{1-\theta}) + \int_1^{y^{1-\theta}} h'(t)\epsilon(t)dt \ll (\log y)^{w-1} .$$

We define $f(k) = \mathcal{R}_k(1 + i\alpha)/\phi(k)$ where $\mathcal{R}_k(s)$ is given by (5.3). In the following lemmas we shall study the Dirichlet series

$$(5.15) \qquad Z(s, \alpha) = \sum_{k \geq 1} d_r(mk)f(nk)k^{-s} = \sum_{k \geq 1} \frac{d_r(mk)\mathcal{R}_{nk}(1 + i\alpha)}{\phi(nk)k^s} .$$

Since f is multiplicative, it is determined by its values on prime powers. Consequently, we may define f by the rule

$$(5.16) \qquad f(p^a) := (1 + ak_p)p^{-a}$$

where

$$(5.17) \qquad k_p := k_p(\alpha) = (1 - p^{i\alpha})(1 - p^{-1-i\alpha})/(1 - p^{-1})$$

which follows from (5.3). Moreover, note that $k_p(0) = 0$.

Lemma 5.11. *Put $l = \log x$ and suppose $|\alpha| \ll (\log x)^{-1}$. For $1 \leq m \leq T$, n squarefree and $n \mid m$ we have*

$$\sum_{k \leq x} d_r(mk)f(nk) = \frac{\sigma_r(m)l^r}{n} \sum_{j=0}^{r} \binom{r}{j} \frac{(-i\alpha \, l)^j}{(r+j)!} + O\left(\frac{d_r(m)j_{\tau_0}(m)l^{r-1}}{n^{1-\epsilon}}\right)$$

where $\tau_0 = 1/3$ is valid and $j_{\tau_0}(m)$ is defined by (3.2).

Proof. This lemma will follow from an application of Perron's formula. However, we begin by analyzing the Dirichlet series $Z(s, \alpha)$. We put $m = \prod_p p^\lambda = uv$ with $u = \prod_{p \mid n} p^\lambda$ and hence by multiplicativity

$$(5.18) \qquad Z(s, \alpha) = \left(\prod_{p^\lambda \| u} \frac{\alpha_p(s, \alpha)}{h_p(s, \alpha)}\right) \prod_{p^\lambda \| v} \left(\frac{\beta_p(s, \alpha)}{h_p(s, \alpha)}\right) \left(\prod_p h_p(s, \alpha)\right)$$

where

$$(5.19) \qquad \alpha_p = \alpha_p(s, \alpha) = \sum_{a \geq 0} d_r(p^{a+\lambda})f(p^{a+1})p^{-as} ,$$

$$(5.20) \qquad \beta_p = \beta_p(s, \alpha) = \sum_{a \geq 0} d_r(p^{a+\lambda})f(p^a)p^{-as} ,$$

$$(5.21) \qquad h_p = h_p(s, \alpha) = \sum_{a \geq 0} d_r(p^a) f(p^a) p^{-as} \ .$$

In the above product we label

$$(5.22) \qquad Z_{11}(s, \alpha) = \prod_{p^\lambda || u} \frac{\alpha_p(s, \alpha)}{h_p(s, \alpha)} \ , \quad Z_{12}(s, \alpha) = \prod_{p^\lambda || v} \frac{\beta_p(s, \alpha)}{h_p(s, \alpha)} \ ,$$

and we set $Z_1(s, \alpha) = Z_{11}(s, \alpha) Z_{12}(s, \alpha)$. Next we remark that the last product factors as

$$(5.23) \qquad \prod_p h_p(s, \alpha) = \frac{\zeta^{2r}(1+s)}{\zeta^r(1+s-i\alpha)} Z_3(s, \alpha) := Z_2(s, \alpha) Z_3(s, \alpha)$$

with $Z_3(s, \alpha)$ holomorphic in $\mathrm{Re}(s) > -1/2$. This shall follow from the expressions we derive for α_p, β_p, and h_p in the next section. Thus we have the factorization

$$(5.24) \qquad Z(s, \alpha) = Z_1(s, \alpha) Z_2(s, \alpha) Z_3(s, \alpha) \ .$$

By Perron's formula we have

(5.25)

$$\sum_{k \leq x} d_r(mk) f(nk) = \frac{1}{2\pi i} \int_{c-iU}^{c+iU} Z(s, \alpha) \frac{x^s}{s} \, ds + O\left(\frac{d_r(m)}{n^{1-\epsilon}} \left(\frac{(\log x)^{2r}}{U} + 1 \right) \right)$$

where $c = (\log x)^{-1}$. Let $\Gamma(U)$ denote the contour consisting of $s \in \mathbb{C}$ such that $\mathrm{Re}(s) = -\frac{\beta}{\log(|\mathrm{Im}(s)|+2)}$ where β is a sufficiently small fixed positive number and $|\mathrm{Im}(s)| \leq U$. Our strategy will be to deform the contour in (5.25) to $\Gamma(U)$, thus picking up the pole at $s = 0$ which shall account for the main term in the lemma. However, we must also bound the contribution coming from $\Gamma(U)$ and the horizontal parts of the contour. In the following section, we shall establish

$$(5.26) \qquad |Z_1(s, \alpha)| \ll \frac{d_r(m) j_{\tau_0}(m)}{n^{1-\epsilon}}$$

in the cases $\mathrm{Re}(s) \geq -1/2$, $|\alpha| \leq cL^{-1}$ and also $\mathrm{Re}(s) \geq -\epsilon$, $|\alpha| \leq \epsilon$. Moreover, we have $|Z_3(s, \alpha)| \ll 1$ in $\mathrm{Re}(s) \geq -1/4$ by the absolute convergence of its series. Furthermore, it is known that

$$\zeta(1+s) - \frac{1}{s} = O(\log(|\mathrm{Im}(s)| + 2)) \quad \text{and} \quad \frac{1}{\zeta(1+s)} = O(\log(|\mathrm{Im}(s)| + 2))$$

on $\Gamma(U)$ and to the right of $\Gamma(U)$. By (5.24) and our previous estimates, we have on $\Gamma(U)$ the bound

$$|Z(s, \alpha)| \ll \log(|\mathrm{Im}(s)| + 2)^{3r} \frac{d_r(m) j_{\tau_0}(m)}{n^{1-\epsilon}} \ .$$

We now deform the above contour to $\Gamma(U)$ picking up the residue at $s = 0$. It follows that

$$\frac{1}{2\pi i}\int_{\Gamma(U)} Z(s,\alpha)\frac{x^s}{s}\,ds \ll \frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}}\int_0^U x^{-\frac{\beta}{\log(|t|+2)}}(\log(t+2))^{3r}\frac{dt}{|t|+1}$$

$$\ll \frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}}(\log U)^{3r+1}\exp\left(-\frac{\beta\log x}{\log(U+2)}\right)$$

$$\ll \frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}}\exp(-\beta_1\sqrt{\log x})$$

by the choice $U = \exp(\beta_2\sqrt{\log x})$ for a suitable β_2. Similarly, we can show that the horizontal edges connecting $\Gamma(U)$ to $[c - iU, c + iU]$ contribute an amount $d_r(m)j_{\tau_0}(m)n^{\epsilon-1}U^{\epsilon-1}$. Collecting estimates we conclude

$$(5.27) \qquad \sum_{k\leq x} d_r(mk)f(nk) = \mathop{res}_{s=0}\left(Z(s,\alpha)x^s s^{-1}\right) + O(d_r(m)j_{\tau_0}(m)n^{\epsilon-1})\ .$$

In the next two subsections we establish the bound (5.26) and in the final subsection we will compute the residue in (5.27).

5.1. **Computing the local factors h_p, α_p, and β_p.** We simplify notation by putting $u = p^{-s-1}$ and $s = \sigma + it$. By (5.16) and (5.21) we have

$$h_p = \sum_{a=0}^{\infty} d_r(p^a)u^a + k_p\sum_{a=0}^{\infty} ad_r(p^a)u^a = (1-u)^{-r-1}(1+(rk_p-1)u)\ .$$

Note that we have use $ad_r(p^a) = rd_{r+1}(p^{a-1})$ for $a \geq 1$. By (5.17), $k_p = 1 - p^{i\alpha} + O(p^{-1+\epsilon})$ and it follows that

$$(5.28) \qquad h_p = (1-p^{-s-1})^{-r-1}\left(1 + \frac{r-1}{p^{s+1}} - \frac{r}{p^{s+1-i\alpha}} + O(p^{-2-\sigma+\epsilon})\right)\ .$$

Equation (5.23) now follows from (5.28). As before we have for $\lambda \geq 1$

$$\beta_p = \sum_{a=0}^{\infty} d_r(p^{a+\lambda})u^a + k_p\sum_{a=0}^{\infty} ad_r(p^{a+\lambda})u^a := \beta + k_p\tilde{\beta}\ .$$

Note that by Lemma 5.4, $\beta = d_r(p^\lambda)(1-u)^{-r}H_{\lambda,r}(u)$ and hence it follows that

$$\beta = d_r(p^\lambda)(1-u)^{-r-1}(1 + O_r(p^{-1-\sigma}))\ .$$

Similarly, we note that $\tilde{\beta} = u\frac{d}{du}(\beta(u))$ from which it follows that

$$\tilde{\beta} = d_r(p^\lambda)u(1-u)^{-r-1}\left((1-u)\frac{d}{du}H_{\lambda,r}(u) - rH_{\lambda,r}(u)\right) \ll d_r(p^\lambda)(1-u)^{-r-1}|u|\ .$$

We conclude that

$$(5.29) \qquad \beta_p = d_r(p^\lambda)(1-u)^{-r-1}\left(1 + O\left(|k_p|p^{-1-\sigma}\right)\right)\ .$$

Likewise, we have

$$\alpha_p = \frac{1}{p}\left(\sum_{a=0}^{\infty} d_r(p^{a+\lambda})u^a + k_p\left(u\sum_{a=0}^{\infty} d_r(p^{a+\lambda})u^a\right)'\right) = \frac{1}{p}\left(\beta(1+k_p) + k_p\tilde{\beta}\right)$$

and it follows from our previous estimates that

$$\text{(5.30)} \qquad \alpha_p = d_r(p^\lambda)p^{-1}(1-u)^{-r-1}O((|k_p|+1)) \ .$$

5.2. Establishing (5.26).

With our estimates for α_p, β_p, and h_p in hand, we are ready to estimate $Z_{1i}(s,\alpha)$. We have by (5.22), (5.21), and (5.30)

$$\text{(5.31)}$$
$$|Z_{11}(s,\alpha)| \leq \prod_{p^\lambda || u} \frac{|\alpha_p(s,\alpha)|}{|h_p(s,\alpha)|} \leq \prod_{p^\lambda || u} \frac{d_r(p^\lambda)(|k_p|+1)}{p}|1+(rk_p-1)p^{-s-1}|^{-1} \ .$$

In addition, by (5.22), (5.21), and (5.29) it follows that

$$\text{(5.32)}$$
$$|Z_{12}(s,\alpha)| \leq \prod_{p^\lambda || v} \frac{|\beta_p(s,\alpha)|}{|h_p(s,\alpha)|} \leq \prod_{p^\lambda || v} d_r(p^\lambda)(1+O(|k_p|p^{-1-\sigma}))|1+(rk_p-1)p^{-s-1}|^{-1} \ .$$

In order to finish bounding these terms, we require a bound for k_p. We shall provide a bound for k_p and hence $Z_{1i}(s,\alpha)$ in each of the cases $0 < |\alpha| \leq cL^{-1}$ and $0 < |\alpha| \leq \epsilon$.

Case 1: $0 < |\alpha| \leq cL^{-1}$ and $\text{Re}(s) \geq -1/2$.
By the definition (5.17) it follows that

$$\text{(5.33)} \qquad |k_p| \ll_c |1-p^{i\alpha}| \ll_c \min(1,\log p/L)$$

since $|p^{i\alpha}| \leq \exp(|\alpha|\log p)$ and $|1-p^{i\alpha}| \leq (|\alpha|\log p)e^{|\alpha|\log p}$. Let $c_1, c_2, \ldots$ be effectively computable constants depending on c and r. We have $|(rk_p-1)p^{-s-1}| \ll p^{-\frac{1}{2}} < 0.5$ if $p \geq c_1$. If $p \leq c_1$ then we may choose T sufficiently large such that (5.33) yields $|k_p| \leq 1/20r$. Thus $|(rk_p-1)p^{-s-1}| \leq 1.1p^{-\frac{1}{2}} < 0.8$ for all primes $p < c_1$ as long as T is sufficiently large. By (5.31) and our aforementioned bounds we obtain,

$$\text{(5.34)} \qquad |Z_{11}(s,\alpha)| \leq \prod_{p^\lambda || u} \frac{c_2 d_r(p^\lambda)}{p} \leq \frac{d_r(u)c_2^{\nu(n)}}{n}$$

where $\nu(n)$ is the number of prime factors of n and

$$\text{(5.35)}$$
$$Z_{12}(s,\alpha) = \prod_{p^\lambda || v} d_r(p^\lambda)\left(\frac{1+O(p^{-\frac{1}{2}+\epsilon})}{1+O(p^{-\frac{1}{2}+\epsilon})}\right) = d_r(v)\prod_{p|v}(1+O(p^{-1/2+\epsilon})) \ .$$

Since $c_2^{\nu(n)} \ll n^\epsilon$ and $Z_1(s,\alpha) = Z_{11}(s,\alpha)Z_{12}(s,\alpha)$ we deduce that $Z_1(s,\alpha) \ll d_r(m)j_{1/3}(v)n^{\epsilon-1}$ in the range $\text{Re}(s) \geq -1/2$ and $|\alpha| \leq cL^{-1}$.

Case 2: $0 < |\alpha| \leq \epsilon$ and $\text{Re}(s) \geq -\epsilon$.
In this case, it follows from (5.17) that

$$\text{(5.36)} \qquad |k_p| \leq 4|1-p^{i\alpha}| \leq \min(8p^\epsilon, 4\epsilon(\log p)p^\epsilon)$$

by employing again the bounds $|p^{i\alpha}| \leq \exp(|\alpha|\log p)$ and $|1-p^{i\alpha}| \leq (|\alpha|\log p)e^{|\alpha|\log p}$. The first bound in (5.36) implies that $|(rk_p-1)p^{-s-1}| \leq (8r+1)p^{-1+2\epsilon} < 0.5$

if p is sufficiently large, say $p > c_3$. If $p \leq c_3$ then $|(rk_p - 1)p^{-s-1}| \leq \frac{4r\epsilon(\log p)}{p^{1-\epsilon}} + \frac{1}{2^{1-\epsilon}} \leq 0.51$ for ϵ sufficiently small. Thus

$$Z_{11}(s,\alpha) = \prod_{p^\lambda||u, p \leq c_3} \left(\frac{c_4 d_r(p^\lambda)}{p^{1-\epsilon}} \right) \prod_{p^\lambda||u, p > c_3} \frac{d_r(p^\lambda)}{p^{1-\epsilon}} (1 + O(p^{-1+\epsilon})) \ll \frac{d_r(u)}{n^{1-\epsilon}} j_{\tau_0}(u)$$

and

$$Z_{12}(s,\alpha) = \prod_{p^\lambda||v, p \leq c_3} (c_5 d_r(p^\lambda)) \prod_{p^\lambda||v, p > c_3} d_r(p^\lambda)(1 + O(p^{-1+2\epsilon})) \ll d_r(v) j_{\tau_0}(v) \ .$$

We conclude that if $\mathrm{Re}(s) \geq -\epsilon$ and $|\alpha| \leq \epsilon$ then $|Z_1(s,\alpha)| \ll d_r(m) j_{\tau_0}(m) n^{\epsilon-1}$. This completes our calculation of (5.26). The lemma will be thus completed once the residue is computed.

5.3. **The residue computation.** We decompose

$$(5.37) \quad Z(s,\alpha)x^s s^{-1} = \zeta(1 + s - i\alpha)^{-r} Z_1(s,\alpha) Z_3(s,\alpha) x^s \zeta(1+s)^{2r} s^{-1} \ .$$

We now compute the Laurent expansion of each factor. We have

$$\zeta^{2r}(1+s)s^{-1} = s^{-2r-1}(1 + a_1 s + a_2 s^2 + \cdots) \ ,$$
$$x^s = 1 + (\log x)s + (\log x)^2 s^2/2! + \cdots \ ,$$
$$\zeta(1 + s - i\alpha)^{-r} = f(-i\alpha) + f'(-i\alpha)s + f^{(2)}(-i\alpha)s^2/2! + \cdots$$

where we put $f(z) = \zeta(1+z)^{-r}$. Note that a simple calculation yields

$$f^{(j)}(-i\alpha) = \begin{cases} r(r-1)\cdots(r-(j-1))(-i\alpha)^{r-j} + O(|\alpha|^{r-j+1}) & 0 \leq j \leq r \\ c_j + O(|\alpha|) & j \geq r+1 \end{cases}$$

and $c_j \in \mathbb{R}$. Next note that $Z_3(s,\alpha)$ has an absolutely convergent power series in $\mathrm{Re}(s) > -1/2$, $|\alpha| \leq cL^{-1}$. It follows that $Z_3(0,\alpha) = Z_3(0,0) + O(|\alpha|) = 1 + O(|\alpha|)$ and $Z_3^{(j)}(0,\alpha) \ll 1$ for $j \geq 0$. Combining these facts yields

$$(5.38) \qquad Z_3(s,\alpha) = (1 + O(|\alpha|)) + O(1)s + O(1)s^2 + \cdots \ .$$

We now compute the Taylor expansion of $Z_1(s,\alpha)$. Since $k_p(0) = 0$ it follows from (5.19), (5.20), (5.21) , and (5.22) that

$$(5.39) \qquad\qquad Z_1(s,0) = \frac{\sigma_r(m, s+1)}{n} \ .$$

By Cauchy's integral formula with a circle of radius $\epsilon/2$, we establish a bound for $Z_1^{(j)}(0,\alpha)$:

$$(5.40) \quad Z_1^{(j)}(0,\alpha) = \frac{1}{2\pi i} \int_{|w-\alpha|=\epsilon/2} \frac{Z_1(0,w)dw}{(w-\alpha)^{j+1}} \ll \left(\frac{2}{\epsilon}\right)^{j+1} \frac{d_r(m) j_{\tau_0}(m)}{n^{1-\epsilon}}$$

by (5.26). By the Taylor series expansion and (5.40) it follows that

$$(5.41) \qquad\qquad Z_1(0,\alpha) = \frac{\sigma_r(m)}{n} + O\left(\frac{d_r(m) j_{\tau_0}(m)}{n^{1-\epsilon}} |\alpha| \right)$$

since $Z_1(0,0) = \sigma_r(m)/n$. Combining (5.40) and (5.41) we obtain
(5.42)

$$Z_1(s,\alpha) = \left(\frac{\sigma_r(m)}{n} + O\left(\frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}}|\alpha| \right) \right) + \sum_{j=0}^{\infty} O(d_r(m)j_{\tau_0}(m)n^{\epsilon-1})s^j/j! \ .$$

We are now in a position to compute the residue. It follows from (5.37), and the above Laurent expansions that the residue at $s = 0$ is

$$res = \sum_{u_1+u_2+u_3+u_4+u_5=2r} \frac{l^{u_1}f^{(u_2)}(-i\alpha)Z_1^{(u_3)}(0,\alpha)Z_3^{(u_4)}(0,\alpha)a_{u_5}}{u_1!u_2!u_3!u_4!} \ .$$

We first show that those terms with $u_5 \geq 1$ contribute a smaller amount. Since $|f^{(u_2)}(-i\alpha)| \ll |\alpha|^{r-u_2}$ for $0 \leq u_2 \leq r$ and $|f^{(u_2)}(-i\alpha)| \ll_r 1$ for $r+1 \leq u_2 \leq 2r$ it follows that the terms with $u_5 \geq 1$ contribute

$$\ll_r \frac{d_r(m)j_\tau(m)}{n^{1-\epsilon}} \sum_{u_1+u_2\leq 2r-1} \frac{l^{u_1}|f^{(u_2)}(-i\alpha)|}{u_1!u_2!}$$

$$\ll \frac{d_r(m)j_\tau(m)}{n^{1-\epsilon}} \left(\sum_{\substack{u_1+u_2\leq 2r-1 \\ 0\leq u_2\leq r}} l^{u_1}|\alpha|^{r-u_2} + \sum_{\substack{u_1+u_2\leq 2r-1 \\ 0\leq u_2\geq r+1}} l^{u_1} \right)$$

$$\ll \frac{d_r(m)j_\tau(m)}{n^{1-\epsilon}}(l^{r-1} + l^{r-2}) \ .$$

We deduce that res equals

$$\sum_{u_1+u_2+u_3+u_4=2r} \frac{l^{u_1}f^{(u_2)}(-i\alpha)Z_1^{(u_3)}(0,\alpha)Z_3^{(u_4)}(0,\alpha)}{u_1!u_2!u_3!u_4!} + O\left(\frac{d_r(m)j_{\tau_0}(m)l^{r-1}}{n^{1-\epsilon}} \right) \ .$$

The contribution from those terms in satisfying $u_1 + u_2 = 2r, u_2 \leq r$ is

$$\left(\sum_{u_1+u_2=2r,u_2\leq r} \frac{l^{u_1}}{u_1!} \frac{f^{(u_2)}(-i\alpha)}{u_2!} \right) \left(\frac{\sigma_r(m)}{n} + O\left(\frac{d_r(m)j_{\tau_0}(v)}{n^{1-\epsilon}}|\alpha| \right) \right) (1 + O(|\alpha|)) \ .$$

$$= \left(\sum_{u_2\leq r} \frac{l^{2r-u_2}}{(2r-u_2)!}\left(\binom{r}{u_2}(-i\alpha)^{r-u_2} + O(|\alpha|^{r-u_2+1}) \right) \right) \left(\frac{\sigma_r(m)}{n} + O\left(\frac{d_r(m)j_{\tau_0}(v)}{n^{1-\epsilon}}|\alpha| \right) \right)$$

$$= \frac{\sigma_r(m)}{n}l^r \sum_{a=0}^{r} \binom{r}{a}\frac{(-i\alpha l)^a}{(r+a)!} + O\left(\frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}} \right) \ .$$

Those terms with $u_1 \leq r-1$ contribute

$$\frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}} \sum_{\substack{u_1+u_2+u_3+u_4=2r \\ u_1\leq r-1}} l^{u_1}|f^{(u_2)}(-i\alpha)| \ll \frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}}l^{r-1}$$

since $|\alpha| \leq cL^{-1} \ll 1$ and the remaining terms are

$$\ll \frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}} \sum_{\substack{u_1+u_2+u_3+u_4=2r \\ u_1+u_2\leq 2r-1,r\leq u_1\leq 2r-1}} l^{u_1}|\alpha|^{u_1+1-r} \ll \frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}}l^{r-1} \ .$$

We thus conclude that

$$res = \frac{\sigma_r(m)l^r}{n} \sum_{a=0}^{r} \binom{r}{a} \frac{(-i\alpha l)^a}{(r+a)!} + O\left(\frac{d_r(m)j_{\tau_0}(m)l^{r-1}}{n^{1-\epsilon}}\right)$$

and the lemma follows by combining this with (5.27).

We have the Taylor series expansion

$$\mathcal{R}_k(1+i\alpha) = \mathcal{R}_k(1) + \mathcal{R}'_k(1)(i\alpha) + \mathcal{R}_k^{(2)}(1)(i\alpha)^2/2 + \cdots .$$

We denote the truncated Taylor series expansion $\mathcal{T}_{k;N}(\alpha) = \sum_{j=0}^{N} \mathcal{R}_k^{(j)}(1)(i\alpha)^j/j!$.

Lemma 5.12. *We have for $l = \log x$, $|\alpha| \ll (\log x)^{-1}$, and $\tau_0 = 1/3$*

$$\sum_{k \leq x} d_r(mk) \frac{\mathcal{T}_{nk;r}(\alpha)}{\phi(nk)} = \frac{\sigma_r(m)l^r}{n} \sum_{j=0}^{r} \binom{r}{j} \frac{(-i\alpha l)^j}{(r+j)!} + O\left(\frac{d_r(m)j_{\tau_0}(v)l^{r-1}}{n^{1-\epsilon}}\right).$$

Proof. We begin by noting that it suffices to prove
(5.43)
$$\sum_{k \leq x} \frac{d_r(mk)\mathcal{R}_{nk}^{(j)}(1)}{\phi(nk)} = \frac{(-1)^j \sigma_r(m)l^{r+j}}{n} \binom{r}{j} \frac{j!}{(r+j)!} + O\left(\frac{d_r(m)j_{\tau_0}(m)l^{r+j-1}}{n}\right).$$

This is since if we multiply the above identity by $(i\alpha)^j/j!$ and sum $j = 0$ to r we obtain the result. The Dirichlet series generating function for the sum in question is

$$i^{-j} \sum_{k=1}^{\infty} \frac{d_r(mk)}{\phi(nk)k^s} \frac{d^j}{d\alpha^j} \mathcal{R}_{nk}(1+i\alpha)\Big|_{\alpha=0} = i^{-j} \frac{d^j}{d\alpha^j} Z(s,\alpha)\Big|_{\alpha=0} .$$

By Perron's formula it follows that the sum in question is

$$\frac{i^{-j}}{2\pi i} \int_{c-iU}^{c+iU} \frac{d^j}{d\alpha^j} Z(s,\alpha)\Big|_{\alpha=0} \frac{x^s}{s}\, ds + O\left(\frac{d_r(m)}{n^{1-\epsilon}} \left(\frac{(\log x)^r}{U} + 1\right)\right)$$

where $c = (\log x)^{-1}$. As in Lemma 5.11 (see the text just after equations (5.25)) we want to deform the contour $[c - iU, c + iU]$ to $\Gamma(U)$ and then pick up the residue at $s = 0$. As this calculation is analogous to the preceding lemma we omit the details. This procedure yields
(5.44)
$$\sum_{k \leq x} \frac{d_r(mk)\mathcal{R}_{nk}^{(j)}(1)}{\phi(nk)} = i^{-j}\mathop{res}_{s=0}\left(\frac{d^j}{d\alpha^j} Z(s,\alpha)\Big|_{\alpha=0} \frac{x^s}{s}\right) + O\left(\frac{d_r(m)j_{\tau_0}(v)l^{r-1}}{n^{1-\epsilon}}\right).$$

Recall that $Z(s,\alpha) = Z_1(s,\alpha)Z_2(s,\alpha)Z_3(s,\alpha)$ where

$$Z_1(s,0) = \frac{\sigma_r(m,s+1)}{n} \ , \ Z_2(s,\alpha) = \frac{\zeta^{2r}(1+s)}{\zeta^r(1+s-i\alpha)} \ , \ Z_3^{(j)}(0,0) \ll 1$$

for all $j \geq 0$. By the product rule we have
(5.45)
$$\frac{d^j}{d\alpha^j} Z(s,\alpha)\Big|_{\alpha=0} = \sum_{u_1+u_2+u_3=j} \binom{j}{u_1,u_2,u_3} Z_1^{(u_1)}(s,0) Z_2^{(u_2)}(s,0) Z_3^{(u_3)}(s,0) .$$

Thus we need to compute

$$\operatorname{res}_{s=0}\left(Z_1^{(u_1)}(s,0)Z_2^{(u_2)}(s,0)Z_3^{(u_3)}(s,0)x^s s^{-1}\right)$$

for all $u_1 + u_2 + u_3 = j$. In fact, it turns out that the main term arises from those triples $(u_1, u_2, u_3) = (0, j, 0)$. We now compute the residue arising from these terms. We have the Laurent expansions,

$$Z_1(s,0) = \frac{\sigma_r(m)}{n} + \frac{\sigma_r^{(1)}(m,1)}{n}s + \cdots ,$$

$$Z_2^{(j)}(s,0) = \frac{r(r-1)\cdots(r-(j-1))(-i)^j}{s^{r+j}} + \frac{c_1}{s^{r+j-1}} + \cdots ,$$

$$Z_3(s,0) = 1 + d_1 s + \cdots .$$

We further remark that by Cauchy's integral formula we may establish $\sigma_r^{(k)}(m,1) \ll d_r(m)j_{\tau_0}(m)$ for some $\tau_0 > 0$. These terms contribute

$$\operatorname*{res}_{s=0} Z_1(s,0)Z_2^{(j)}(s,0)Z_3(s,0)x^s s^{-1} =$$

$$\frac{\sigma_r(m)r(r-1)\cdots(r-(j-1))l^{r+j}(-i)^j}{n(r+j)!} + O\left(\frac{d_r(m)j_{\tau_0}(m)l^{r+j-1}}{n}\right) .$$

A similar calculation shows that for those triples (u_1, u_2, u_3) such that $u_2 \leq j-1$ then

$$\operatorname*{res}_{s=0} Z_1^{(u_1)}(s,0)Z_2^{(u_2)}(s,0)Z_3^{(u_3)}(s,0)x^s s^{-1} \ll \frac{d_r(m)j_{\tau_0}(m)(\log x)^{r+j-1}}{n} .$$

By combining the last two expressions with (5.44) and (5.45) completes the lemma.

We deduce the following corollary to Lemmas 5.11 and 5.12:

Lemma 5.13.

$$\sum_{k\leq x} d_r(mk)\left(f(nk) - \frac{\mathcal{I}_{nk,r}(\alpha)}{\phi(nk)}\right) \ll |\alpha|^{r+1}L^{2r}\frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}} .$$

Proof. Note that $f(nk) = \frac{\mathcal{I}_{nk,r}(\alpha)}{\phi(nk)} + \alpha^{r+1}g(\alpha; nk)$ where g is entire in α. Moreover, it follows that

$$\sum_{k\leq x} d_r(mk)\left(f(nk) - \frac{\mathcal{I}_{nk,r}(\alpha)}{\phi(nk)}\right) = \alpha^{r+1}g^*(\alpha; n, x)$$

where g^* entire in α. Combining Lemmas 5.11 and 5.12 we deduce that

$$\max_{|\alpha|\leq cL^{-1}} |\alpha^{r+1}g^*(\alpha; n, x)| \ll \frac{d_r(m)j_{\tau_0}(m)L^{r-1}}{n^{1-\epsilon}}$$

and hence by the maximum modulus principle

$$\max_{|\alpha|\leq cL^{-1}} |g^*(\alpha; n, x)| \ll \frac{d_r(m)j_{\tau_0}(m)L^{2r}}{n^{1-\epsilon}}$$

which implies the lemma.

6. Proof of Theorem 2

In this section we apply the lemmas to manipulate I into a suitable form for evaluation. Recall that by (4.7)

$$(6.1) \qquad I = \sum_{k \leq y} \frac{d_r(k)P([k]_y)}{k} \sum_{j \leq \frac{kT}{2\pi}} b(j)e(-j/k) + O_\epsilon(yT^{\frac{1}{2}+\epsilon}) \ .$$

By Perron's formula with $c = 1 + L^{-1}$ the inner sum is

$$\sum_{j \leq \frac{kT}{2\pi}} b(j)e(-j/k) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \mathcal{Q}^*(s, \alpha, k) \left(\frac{kT}{2\pi}\right)^s \frac{ds}{s} + O(kT^\epsilon)$$

where $\mathcal{Q}^*(s, \alpha, k) = \sum_{j=1}^{\infty} b(j)j^{-s}e(-j/k)$. Pulling the contour left to $c_0 = 1/2 + L^{-1}$ we obtain

$$(6.2) \qquad \begin{aligned} \sum_{j \leq \frac{kT}{2\pi}} b(j)e(-j/k) &= R_1 + R_{1+i\alpha} \\ &+ \frac{1}{2\pi i}\left(\int_{c-iT}^{c_0-iT} + \int_{c_0-iT}^{c_0+iT} + \int_{c_0+iT}^{c+iT}\right) \mathcal{Q}^*(s, \alpha, k)\left(\frac{kT}{2\pi}\right)^s \frac{ds}{s} \end{aligned}$$

where R_u is the residue at $s = u$. By Lemma 5.6 the left and horizontal edges contribute $yT^{1/2+\epsilon}$. Moreover by (4.6) it follows that

$$(6.3) \qquad \mathcal{Q}^*(s, \alpha, k) = \sum_{h \leq y} \frac{d_r(h)P([h]_y)\mathcal{Q}(s, \alpha, h/k)}{h^s}$$

where $\mathcal{Q}(s, \alpha, h/k)$ is defined by (5.1). We will now invoke Lemma 5.3, however we require that h, k be relatively prime. Therefore we set $\frac{h}{k} = \frac{H}{K}$ where $H = h/(h,k)$, $K = k/(h,k)$, and $(H, K) = 1$. We deduce

$$R_1 = \sum_{h \leq y} d_r(h)P([h]_y) \operatorname*{res}_{s=1}\left(\mathcal{Q}(s, \alpha, H/K)\left(\frac{TK}{2\pi H}\right)^s s^{-1}\right) \ .$$

By an application of Lemma 5.3(i) this is

$$(6.4)$$
$$\begin{aligned} R_1 &= K\sum_{h \leq y} d_r(h)P([h]_y) \operatorname*{res}_{s=1}\left(\zeta^2(s)\left(\frac{\zeta'}{\zeta}(s-i\alpha) - \mathcal{G}(s, \alpha, K)\right)\left(\frac{T}{2\pi HK}\right)^s s^{-1}\right) \\ &= \frac{T}{2\pi}\sum_{h \leq y} \frac{d_r(h)P([h]_y)}{H} \\ &\quad \cdot \left(\left((\zeta'/\zeta)(\overline{\tau}) - \mathcal{G}(1, \alpha, K)\right)\log\left(\frac{Te^{2\gamma-1}}{2\pi HK}\right) + \left((\zeta'/\zeta)'(\overline{\tau}) - \mathcal{G}'(1, \alpha, K)\right)\right) \end{aligned}$$

where we put $\tau = 1 + i\alpha$. Likewise Lemma 5.3(ii) implies

(6.5)
$$R_{1+i\alpha} = \sum_{h \leq y} d_r(h)P([h]_y) \operatorname*{res}_{s=\tau} \left(\mathcal{Q}(s, \alpha, H/K) \left(\frac{TK}{2\pi H} \right)^s s^{-1} \right)$$

$$= -\frac{T}{2\pi} \frac{\zeta^2(\tau)}{\tau} \sum_{h \leq y} \frac{d_r(h)P([h]_y)}{H} \left(\frac{T}{2\pi H} \right)^{i\alpha} \frac{K\mathcal{R}_K(\tau)}{\phi(K)} \, .$$

Combining (6.1), (6.2), (6.4), and (6.5) we deduce

$$I = \frac{T}{2\pi} \sum_{h,k \leq y} \frac{d_r(h)d_r(k)P([h]_y)P([k]_y)(h,k)}{hk} \left(\log \frac{Te^{2\gamma-1}}{2\pi HK} \left((\zeta'/\zeta)(\tau) - \mathcal{G}(1, \alpha, K) \right) \right.$$

$$\left. + \left(\zeta'/\zeta \right)'(\tau) - \mathcal{G}'(1, \alpha, K) - \frac{\zeta^2(\tau)}{\tau} \left(\frac{T}{2\pi H} \right)^{i\alpha} \frac{K\mathcal{R}_K(\tau)}{\phi(K)} \right) + O(yT^{1/2+\epsilon})$$

where $\mathcal{G}(s, \alpha, K)$ is defined by (5.2). We may write for $j = 0, 1$ $\mathcal{G}^{(j)}(1, \alpha, K) = \sum_{p|K} p^{i\alpha} \log^{j+1} p + O(C_j(K))$. By Lemma 5.7, the $O(C_j(K))$ terms contribute $O(TL^{(r+1)^2})$. Whence

$$I = \frac{T}{2\pi} \sum_{h,k \leq y} \frac{d_r(h)d_r(k)P([h]_y)P([k]_y)(h,k)}{hk} \left(\log \frac{Te^{2\gamma-1}}{2\pi HK} \left((\zeta'/\zeta)(\overline{\tau}) - \sum_{p|K} p^{i\alpha} \log p \right) \right.$$

$$\left. + (\zeta'/\zeta)'(\overline{\tau}) - \sum_{p|K} p^{i\alpha} \log^2 p - \frac{\zeta^2(\tau)}{\tau} \left(\frac{T}{2\pi H} \right)^{i\alpha} \frac{K\mathcal{R}_K(\tau)}{\phi(K)} \right) + O(yT^{1/2+\epsilon})$$

where $z = 1 + i\alpha$. Insertion of the identity

$$f((h,k)) = \sum_{\substack{m|h \\ m|k}} \sum_{n|m} \mu(n) f\left(\frac{m}{n} \right)$$

produces

$$I = \frac{T}{2\pi} \sum_{h,k \leq y} \frac{d_r(h)P([h]_y)d_r(k)P([k]_k)}{hk} \sum_{\substack{m|h \\ m|k}} m \sum_{n|m} \frac{\mu(n)}{n}$$

$$\cdot \left(\log \frac{Te^{2\gamma-1}m^2}{2\pi hkn^2} \left((\zeta'/\zeta)(\overline{\tau}) - \sum_{p|\frac{nk}{m}} p^{i\alpha} \log p \right) + \left(\zeta'/\zeta \right)'(\overline{\tau}) \right.$$

$$\left. - \sum_{p|\frac{nk}{m}} p^{i\alpha} \log^2 p - \frac{\zeta^2(\tau)}{\tau} \left(\frac{Tm}{2\pi nh} \right)^{i\alpha} \left(\frac{nk}{m} \right) \frac{\mathcal{R}_{\frac{nk}{m}}(\tau)}{\phi(\frac{nk}{m})} \right) + O(yT^{1/2+\epsilon}) \, .$$

Changing summation order and making the variable changes $h \to hm$ and $k \to km$ yields

$$I = \frac{T}{2\pi} \sum_{m \le y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \le \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk}$$

$$\cdot \left(\log \frac{Te^{2\gamma-1}}{2\pi hkn^2} \left((\zeta'/\zeta)(\overline{\tau}) - \sum_{p|nk} p^{i\alpha} \log p \right) + \left(\zeta'/\zeta \right)' (\overline{\tau}) \right.$$

$$\left. - \sum_{p|nk} p^{i\alpha} \log^2 p - \frac{\zeta^2(\tau)}{\tau} \left(\frac{T}{2\pi nh} \right)^{i\alpha} \frac{nk\mathcal{R}_{nk}(\tau)}{\phi(nk)} \right) + O(yT^{1/2+\epsilon}) \ .$$

Rearrange this as $I = I_1 + I_2 + O(yT^{1/2+\epsilon})$ where

$$I_1 = \frac{T}{2\pi} \sum_{m \le y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \le \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk}$$

$$\cdot \left(-\log \frac{Te^{2\gamma-1}}{2\pi hkn^2} \sum_{p|nk} p^{i\alpha} \log p - \sum_{p|nk} p^{i\alpha} \log^2 p \right) \ ,$$

(6.6)

$$I_2 = \frac{T}{2\pi} \sum_{m \le y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \le \frac{y}{m}} \frac{d_r(mh)P([mk]_y)d_r(mk)P([mk]_y)}{hk}$$

$$\cdot \left(\log \left(\frac{Te^{2\gamma-1}}{2\pi hkn^2} \right) \frac{\zeta'}{\zeta}(\overline{\tau}) + \left(\frac{\zeta'}{\zeta} \right)' (\overline{\tau}) - \frac{\zeta^2(\tau)}{\tau} \left(\frac{T}{2\pi nh} \right)^{i\alpha} \frac{nk\mathcal{R}_{nk}(\tau)}{\phi(nk)} \right) \ .$$

The first sum is

$$I_1 = \frac{T}{2\pi} \sum_{m \le y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \le \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk}$$

$$\cdot \left(-\log \frac{T}{2\pi hk} \sum_{p|k} p^{i\alpha} \log p - \sum_{p|k} p^{i\alpha} \log^2 p + O(L \log n) \right) \ .$$

A calculation shows that the $O(L \log n)$ contributes $O(TL^{(r+1)^2})$. Since $\phi(m)m^{-1} = \sum_{n|m} \mu(n)n^{-1}$ we deduce that

(6.7)

$$I_1 = \frac{T}{2\pi} \sum_{m \le y} \frac{\phi(m)}{m^2} \sum_{h,k \le \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk}$$

$$\cdot \left(-\log \frac{T}{2\pi hk} \sum_{p|k} p^{i\alpha} \log p - \sum_{p|k} p^{i\alpha} \log^2 p \right) + O(TL^{(r+1)^2}) \ .$$

This puts I_1 in a suitable form to be evaluated by the lemmas. We now simplify I_2 by substituting the Laurent expansions

$$(\zeta'/\zeta)(\overline{\tau}) = (i\alpha)^{-1} + O(1) \ ,$$

$$\left(\zeta'/\zeta\right)'(\overline{\tau}) = (i\alpha)^{-2} + O(1) \ ,$$

$$\zeta^2(\tau)\tau^{-1} = (i\alpha)^{-2} + (2\gamma - 1)(i\alpha)^{-1} + O(1)$$

in (6.6). The $O(1)$ terms of these Laurent expansions contribute

$$TL \sum_{m \leq y} \frac{d_r(m)^2}{m} \sum_{n|m} \frac{1}{n} \sum_{h,k \leq \frac{y}{m}} \frac{d_r(h)d_r(k)}{hk} \ll TL^{(r+1)^2}$$

by (3.3) and

$$T \sum_{m \leq y} \frac{d_r(m)}{m} \sum_{n|m} 1 \sum_{h \leq \frac{y}{m}} \frac{d_r(h)}{h} \left| \sum_{k \leq \frac{y}{m}} d_r(mk)f(nk) \right|$$

$$\ll TL^r \sum_{m \leq y} \frac{d_r(m)}{m} \sum_{n|m} \frac{d_r(m)\sigma_r(m)L^r}{n^{1-\epsilon}} \ll TL^{(r+1)^2-1}$$

by (3.3) and Lemma 5.11. Thus we deduce

$$I_2 = \frac{T}{2\pi} \sum_{m \leq y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk}$$

$$\cdot \left(\frac{1 + i\alpha \log \frac{T}{2\pi hkn^2} - \left(\frac{T}{2\pi hn}\right)^{i\alpha} \frac{nk\mathcal{R}_{nk}(\tau)}{\phi(nk)}}{(i\alpha)^2} \right)$$

plus an error term $O(TL^{(r+1)^2})$. By Lemma 5.13 we may replace $\frac{\mathcal{R}_{nk}(\tau)}{\phi(nk)}$ by $\frac{\mathcal{T}_{nk;r}(\alpha)}{\phi(nk)}$ at the expense of the error

$$\ll |\alpha|^{-2} TL^r \sum_{m \leq y} \frac{d_r(m)}{m} \sum_{n|m} |\alpha|^{r+1} L^{2r} \frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}} \ll TL^{(r+1)^2} \ .$$

Therefore we have

$$I_2 = \frac{T}{2\pi} \sum_{m \leq y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk}$$

$$\cdot \left(1 + i\alpha \log \frac{T}{2\pi hkn^2} - \left(\frac{T}{2\pi hn}\right)^{i\alpha} \frac{nk\mathcal{T}_{nk;r}(\alpha)}{\phi(nk)} \right) (i\alpha)^{-2} + O(TL^{(r+1)^2}) \ .$$

A calculation shows that $\mathcal{R}_k(1) = \phi(k)/k$, $\mathcal{R}'_k(1) = -\phi(k)\log k/k$ and thus it follows that

$$\frac{\mathcal{T}_{nk;r}(\alpha)}{\phi(nk)} = \frac{1}{nk} \left(1 - \log(nk)(i\alpha) \right) + \sum_{j=2}^{r} \frac{\mathcal{R}_{nk}^{(j)}(1)(i\alpha)^j}{\phi(nk)j!} \ .$$

We further decompose $I_2 = I_{21} + I_{22} + O(TL^{(r+1)^2})$ where

$$
\begin{aligned}
I_{21} = \frac{T}{2\pi} \sum_{m \le y} \frac{1}{m} \sum_{n \mid m} \frac{\mu(n)}{n} \sum_{h,k \le \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk} \\
\cdot \left(\frac{1 + i\alpha \log \frac{T}{2\pi hkn^2} - \left(\frac{T}{2\pi hn}\right)^{i\alpha} (1 - (i\alpha)\log(nk))}{(i\alpha)^2} \right)
\end{aligned}
$$
(6.8)

(6.9)

and
$$
\begin{aligned}
I_{22} = -\frac{T}{2\pi} \sum_{j=2}^{r} \frac{(i\alpha)^{j-2}}{j!} \sum_{m \le y} \frac{1}{m} \sum_{n \mid m} \frac{\mu(n)}{n} \\
\cdot \sum_{h,k \le \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk} \left(\frac{T}{2\pi hn}\right)^{i\alpha} \frac{nk\mathcal{R}_{nk}^{(j)}(1)}{\phi(nk)} .
\end{aligned}
$$

6.1. Evaluation of I_1. By (6.7) it follows that

$$
I_1 = \frac{T}{2\pi} \left(-La_{0,0,1} + a_{1,0,1} + a_{0,1,1} - a_{0,0,2}\right) + O(TL^{(r+1)^2})
$$
(6.10)

where for $u, v, w \in \mathbb{Z}_{\ge 0}$ we define $a_{u,v,w}$ to be the sum

$$
\sum_{mh,mk \le y} \frac{\phi(m)d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)(\log h)^u (\log k)^v}{m^2 hk} \sum_{p \mid k} p^{i\alpha}(\log p)^w .
$$

By (6.10) it suffices to evaluate $a_{u,v,w}$. Inverting summation we have

$$
\begin{aligned}
a_{u,v,w} = \sum_{m \le y} \frac{\phi(m)}{m^2} \sum_{p \le \frac{y}{m}} \frac{p^{i\alpha}(\log p)^w}{p} \left(\sum_{h \le \frac{y}{m}} \frac{d_r(mh)P([mh]_y)(\log h)^u}{h} \right) \\
\cdot \left(\sum_{k \le \frac{y}{pm}} \frac{d_r(mpk)P([mpk]_y)(\log pk)^v}{k} \right)
\end{aligned}
$$

By (i) and (ii) of Lemma 5.8 we have

$$
\begin{aligned}
a_{u,v,w} = \frac{1}{(r-1)!^2} \sum_{mp \le y} \frac{\phi(m)\sigma_r(m)^2 p^{i\alpha}(\log p)^w}{m^2 p} \log\left(\frac{y}{m}\right)^{r+u} \log\left(\frac{y}{pm}\right)^r \\
\cdot \int_0^1 \int_0^1 F_1(\theta_1, m)F_2(\theta_2, pm)\, d\theta_1 d\theta_2 + \epsilon_1 + \epsilon_2 + \epsilon_3
\end{aligned}
$$

where

$$
\epsilon_1 \ll \sum_{m \le y} \frac{\sigma_r(m)L^{u+r}}{m} \sum_{p \le y} \frac{(\log p)^w j_{\tau_0}(m)d_r(m)L^{v+r-1}}{p} ,
$$

$$
\epsilon_2 \ll \sum_{m \le y} \frac{j_{\tau_0}(m)d_r(m)L^{u+r-1}}{m} \sum_{p \le y} \frac{(\log p)^w \sigma_r(pm)L^{v+r}}{p} ,
$$

$$
\epsilon_3 \ll \sum_{m \le y} \frac{j_{\tau_0}(m)d_r(m)L^{u+r-1}}{m} \sum_{p \le y} \frac{(\log p)^w j_{\tau_0}(m)d_r(m)L^{r-1}}{p} .
$$

By (5.13) it follows that

$$\epsilon_1 \ll L^{u+v+w+2r-1} \sum_{m \leq y} \frac{d_r(m)^2 j_1(m) j_{\tau_0}(m)}{m} \ll L^{u+v+w+r^2+2r-1} \ .$$

A similar calculation gives $\epsilon_2, \epsilon_3 \ll L^{u+v+w+r^2+2r-1}$. Recalling (5.14) and rearranging a little, yields

$$a_{u,v,w} = \frac{(\log y)^{2r+u+v}}{((r-1)!)^2} \int_0^1 \int_0^1 \theta_1^{r+u-1} \theta_2^{r-1} \sum_{p \leq y} \frac{p^{i\alpha}(\log p)^w}{p}$$

(6.11)

$$\cdot \sum_{m \leq \frac{y}{p}} \frac{\phi(m)\sigma_r(m)\sigma_r(pm)}{m^2} g_{u,v}\left([m]_y, [p]_y\right) d\theta_1 d\theta_2 + O(L^{\max(u,v)+r^2+r})$$

where

(6.12) $g_{u,v}(\delta, \beta) =$
$$(1-\delta)^{r+u}(1-\beta-\delta)^r \left(\beta + \theta_2(1 - \beta - \delta)\right)^v P(\delta+\theta_1(1-\delta))P(\delta+\beta+\theta_2(1-\beta-\delta)) \ .$$

By Lemma 5.9 (ii), (6.11) becomes

$$a_{u,v,w} = rC_r(\log y)^{r^2+2r+u+v} \int_0^1 \int_0^1 \theta_1^{r+u-1} \theta_2^{r-1} \sum_{p \leq y} \frac{p^{i\alpha}(\log p)^w}{p}$$

$$\cdot \int_0^{1-[p]_y} \delta^{r^2-1} g_{u,v}\left(\delta, [p]_y\right) d\delta \ d\theta_1 d\theta_2 + \epsilon_4 + O(L^{\max(u,v)+r^2+r})$$

where C_r is defined by (1.11) and

$$\epsilon_4 \ll L^{2r+u+v} \sum_{p \leq y} \frac{(\log p)^w}{p}(L^{r^2}p^{-1} + L^{r^2-1}) \ll L^{r^2+2r+u+v+w-1}$$

since $w \geq 1$. Inverting summation

$$a_{u,v,w} = rC_r(\log y)^{r^2+2r+u+v} \int_0^1 \int_0^1 \int_0^1 \theta_1^{r+u-1} \theta_2^{r-1} \delta^{r^2-1}$$

$$\cdot \sum_{p \leq y^{1-\delta}} \frac{p^{i\alpha}(\log p)^w}{p} g_{u,v}\left(\delta, [p]_y\right) d\delta d\theta_1 d\theta_2 + O(L^{r^2+2r+u+v+w-1}) \ .$$

An application of Lemma 5.10 yields

$$a_{u,v,w} = rC_r(\log y)^{r^2+2r+u+v+w} \sum_{j=0}^{\infty} \frac{(i\alpha \log y)^j}{j!}$$

$$\cdot \int_0^1 \int_0^1 \int_0^1 \int_0^{1-\delta} \theta_1^{r+u-1} \theta_2^{r-1} \delta^{r^2-1} \beta^{j+w-1} g_{u,v}(\delta, \beta) d\beta d\delta d\theta_1 d\theta_2 (1 + O(L^{-1})) \ .$$

We write

$$\int_0^1 \int_0^1 \theta_1^{r+u-1} \theta_2^{r-1} g_{u,v}(\delta, \beta) d\theta_1 d\theta_2 = (1-\delta)^{r+u}(1-\beta-\delta)^r Q_{r+u-1}(\delta) R_v(\delta, \beta)$$

where
$$Q_{r+u-1}(\delta) = \int_0^1 \theta_1^{r+u-1} P(\delta + \theta_1(1-\delta)) \, d\theta_1 \; ,$$

$$R_v(\delta, \beta) = \int_0^1 \theta_2^{r-1} (\beta + \theta_2(1-\delta-\beta))^v P(\delta + \beta + \theta_2(1-\delta-\beta)) \, d\theta_2$$

and hence

$$a_{u,v,w} \sim rC_r (\log y)^{r^2+2r+u+v+w} \sum_{j=0}^{\infty} \frac{(i\alpha \log y)^j}{j!}$$

$$\cdot \int_0^1 \delta^{r^2-1}(1-\delta)^{r+u} Q_{u+r-1}(\delta) \int_0^{1-\delta} \beta^{j+w-1}(1-\beta-\delta)^r R_v(\delta, \beta) \, d\beta d\delta \; .$$

Now note that

$$R_0(\delta, \beta) = Q_{r-1}(\delta+\beta) \; , \quad R_1(\delta, \beta) = \beta Q_{r-1}(\delta+\beta) + (1-\delta-\beta)Q_r(\delta+\beta) \; .$$

We see that

$$a_{u,0,w} \sim rC_r (\log y)^{r^2+2r+u+w} \sum_{j=0}^{\infty} \frac{(i\alpha \log y)^j}{j!}$$

$$\cdot \int_0^1 \int_0^{1-\delta} \delta^{r^2-1}(1-\delta)^{r+u}(1-\beta-\delta)^r \beta^{j+w-1} Q_{u+r-1}(\delta) Q_{r-1}(\delta+\beta) \, d\beta d\delta$$

and

$$a_{u,1,w} \sim rC_r (\log y)^{r^2+2r+1+u+w} \sum_{j=0}^{\infty} \frac{(i\alpha \log y)^j}{j!}$$

$$\cdot \left(\int_0^1 \int_0^{1-\delta} \delta^{r^2-1}(1-\delta)^{r+u}(1-\beta-\delta)^r \beta^{j+w} Q_{u+r-1}(\delta) Q_{r-1}(\delta+\beta) \, d\beta d\delta + \right.$$

$$\left. \int_0^1 \int_0^{1-\delta} \delta^{r^2-1}(1-\delta)^{r+u}(1-\beta-\delta)^{r+1} \beta^{j+w-1} Q_{u+r-1}(\delta) Q_r(\delta+\beta) \, d\beta d\delta \right) \; .$$

For $\vec{n} = (n_1, n_2, n_3, n_4, n_5) \in (\mathbb{Z}_{\geq 0})^5$ we recall the definition (1.9)

$$i_P(\vec{n}) = \int_0^1 \int_0^{1-x_1} x_1^{r^2-1}(1-x_1)^{n_1}(1-x_1-x_2)^{n_2} x_2^{n_3} Q_{n_4}(x_1) Q_{n_5}(x_1+x_2) \, dx_2 dx_1$$

and hence

$$a_{0,0,1} = rC_r L^{(r+1)^2} \sum_{j=0}^{\infty} \frac{z^j \eta^{j+(r+1)^2}}{j!} i_P(r, r, j, r-1, r-1) \; ,$$

$$a_{1,0,1} = rC_r L^{(r+1)^2+1} \sum_{j=0}^{\infty} \frac{z^j \eta^{j+(r+1)^2+1}}{j!} i_P(r+1, r, j, r, r-1) \; ,$$

$$a_{0,0,2} = rC_r L^{(r+1)^2+1} \sum_{j=0}^{\infty} \frac{z^j \eta^{j+(r+1)^2+1}}{j!} i_P(r, r, j+1, r-1, r-1) \; ,$$

$$a_{1,0,1} = rC_r L^{(r+1)^2+1} \sum_{j=0}^{\infty} \frac{z^j \eta^{j+(r+1)^2+1}}{j!} i_P(r, r, j+1, r-1, r-1) + i_P(r, r+1, j, r-1, r) \; .$$

Combining these identities with (6.10) we arrive at

(6.13)

$$I_1 \sim rC_r \frac{T}{2\pi} L^{(r+1)^2+1} \sum_{j=0}^{\infty} \frac{z^j \eta^{j+(r+1)^2}}{j!}$$

$$\cdot \left(-i_P(r,r,j,r-1,r-1) + \eta(i_P(r+1,r,j,r,r-1) + i_P(r,r+1,j,r-1,r)) \right)$$

and this is valid up to an error which is smaller by a factor $O(L^{-1})$.

6.2. **Evaluation of I_{21}.** We recall that

$$I_{21} \sim \frac{T}{2\pi} \sum_{m \leq y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk}$$

$$\cdot \left(\frac{1 + i\alpha \log \frac{T}{2\pi hkn^2} - \left(\frac{T}{2\pi hn}\right)^{i\alpha}(1-(i\alpha)\log nk)}{(i\alpha)^2} \right) .$$

A little algebra shows that the expression within the brackets simplifies to

$$\log\left(\frac{T}{2\pi hn}\right)\log(nk) - (1 - (i\alpha)\log nk)\log\left(\frac{T}{2\pi hn}\right)^2 \sum_{j=0}^{\infty} \frac{(i\alpha \log\left(\frac{T}{2\pi hn}\right))^j}{(j+2)!} .$$

We may replace $\log\frac{T}{2\pi hn}$ by $\log\frac{T}{2\pi h}$ and $\log(nk)$ by $\log k$ up to an error of $L(\log n)$. This error term contributes $O(TL^{(r+1)^2})$ as long as we use $|\alpha| \leq cL^{-1}$. It thus follows that

$$I_{21} \sim \frac{T}{2\pi} \sum_{m \leq y} \frac{\phi(m)}{m^2} \sum_{h,k \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk}$$

$$\cdot \left(\log\left(\frac{T}{2\pi h}\right)\log k - (1 - (i\alpha)\log k)\log\left(\frac{T}{2\pi h}\right)^2 \sum_{j=0}^{\infty} \frac{(i\alpha \log\left(\frac{T}{2\pi h}\right))^j}{(j+2)!} \right)$$

and hence

(6.14) $$I_2 \sim \frac{T}{2\pi} \left(b_{1,1} - \sum_{j=0}^{\infty} \frac{(i\alpha)^j}{(j+2)!} b_{j+2,0} + \sum_{j=0}^{\infty} \frac{(i\alpha)^{j+1}}{(j+2)!} b_{j+2,1} \right)$$

where

$$b_{u,v} = \sum_{m \leq y} \frac{\phi(m)}{m^2} \sum_{h,k \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y)(\log \frac{T}{2\pi h})^u d_r(mk)P([mk]_y)(\log k)^v}{hk}$$

for $u, v \geq 0$. By parts (iii) and (i) of Lemma 5.8 it follows that $b_{u,v}$ is asymptotic to

$$\frac{(\log y)^{u+r}}{((r-1)!)^2} \sum_{m \leq y} \frac{\phi(m)\sigma_r(m)^2}{m^2} \log\left(\frac{y}{m}\right)^{r+v} \int_0^{1-[m]_y} F_3(\theta_1, m)d\theta_1 \int_0^1 F_1(\theta_2, m)d\theta_2$$

where F_1, F_3 are given by (5.14). This is valid up to an error $O(L^{r^2+2r+u+v-1})$. Next we exchange summation order and we recall the definitions for the F_i (5.14) to obtain $b_{u,v}$ is asymptotic to

$$\frac{(\log y)^{2r+u+v}}{((r-1)!)^2} \int_0^1 \int_0^1 \theta_1^{r-1} (\eta^{-1}-\theta_1)^u \theta_2^{r+v-1} \sum_{m \leq y^{1-\theta_1}} \frac{\phi(m)\sigma_r(m)^2}{m^2} g\left([m]_y\right) d\theta_1 d\theta_2$$

where $g(\delta) = (1-\delta)^{r+v} P(\delta+\theta_1) P(\delta+(1-\delta)\theta_2)$. By Lemma 5.9(ii) we have $b_{u,v}$ equals

$$C_r (\log y)^{r^2+2r+u+v} \int_0^1 \int_0^1 \int_0^{1-\theta_1} \theta_1^{r-1} (\eta^{-1}-\theta_1)^u \theta_2^{r+v-1} \delta^{r^2-1} g(\delta) d\delta d\theta_1 d\theta_2$$

plus an error $O(L^{r^2+r+\max(u,v)})$. Since $Q_{r+v-1}(\delta) = \int_0^1 \theta_2^{r+v-1} P(\delta+(1-\delta)\theta_2) d\theta_2$ it follows that

$$b_{u,v} \sim C_r (\log y)^{r^2+2r+u+v} k_P(u, r+v, r+v-1)$$

where we recall (1.10)

$$k_P(n_1, n_2, n_3) = \int_0^1 \int_0^{1-\theta_1} \theta_1^{r-1} (\eta^{-1}-\theta_1)^{n_1} \delta^{r^2-1} (1-\delta)^{n_2} P(\theta_1+\delta) Q_{n_3}(\delta) d\delta d\theta_1 .$$

We conclude

(6.15)
$$I_{21} \sim C_r \frac{T}{2\pi} (\log y)^{r^2+2r+2} \sum_{j=0}^{\infty} (z\eta)^j \left(\frac{k_P(j+1, r+1, r)}{(j+1)!} - \frac{k_P(j+2, r, r-1)}{(j+2)!} \right) .$$

It can be checked that the error term $O(L^{r^2+r+\max(u,v)})$ contributes an amount $O(L^{-1})$ smaller than the main term.

6.3. Evaluation of I_{22}. By (6.9)

(6.16)
$$I_{22} = -\frac{T}{2\pi} \sum_{j=2}^{r} \frac{(i\alpha)^{j-2}}{j!} \sum_{u=0}^{\infty} \frac{(i\alpha)^u}{u!} c_{u,j}$$

where

(6.17)
$$c_{u,j} = \sum_{m \leq y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n^{i\alpha}} \left(\sum_{h \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y)}{h} \log\left(\frac{T}{2\pi h}\right)^u \right)$$
$$\cdot \left(\sum_{k \leq \frac{y}{m}} d_r(mk)P([mk]_y) \frac{\mathcal{R}_{nk}^{(j)}(1)}{\phi(nk)} \right) .$$

Applying partial summation to (5.43) yields

(6.18)

$$
\sum_{k \le \frac{y}{m}} d_r(mk) P([mk]_y) \frac{\mathcal{R}_{nk}^{(j)}(1)}{\phi(nk)} =
$$

$$
\frac{\sigma_r(m)(-1)^j j! \binom{r}{j} \log\left(\frac{y}{m}\right)^{r+j}}{n(r+j-1)!} \int_0^1 \theta^{r+j-1} P([m]_y + (1 - [m]_y)\theta) d\theta + O(E(y))
$$

where $E(y)$ denotes the error term in (5.43). We apply Lemma 5.8(iii) to the first factor in (6.17) and we apply (6.18) to the second factor of (6.17) to obtain

$$
c_{u,j} = \frac{(-1)^j j! \binom{r}{j} (\log y)^{u+r}}{(r-1)!(r+j-1)!} \sum_{m \le y} \sigma_r(m)^2 \log\left(\frac{y}{m}\right)^{r+j} \left(\sum_{n|m} \frac{\mu(n)}{n^{1+i\alpha}} \right)
$$

$$
\cdot \int_0^{1-[m]_y} F_3(\theta_1, m) d\theta_1 \int_0^1 \theta_2^{r+j-1} P([m]_y + (1 - [m]_y)\theta_2) \, d\theta_2
$$

where $F_3(\theta_1, m) = \theta_1^{r-1}(\eta^{-1} - \theta_1)^u P([m]_y + \theta_1)$. Further simplification gives

$$
c_{u,j} = \frac{(-1)^j j! \binom{r}{j} (\log y)^{2r+u+j}}{(r-1)!(r+j-1)!} \int_0^1 \int_0^1 \theta_1^{r-1} \theta_2^{r+j-1} (\eta^{-1} - \theta_1)^u \sum_{m \le y^{1-\theta_1}} \frac{\sigma_r(m)^2}{m}
$$

$$
\cdot \sum_{n|m} \frac{\mu(n)}{n^{1+i\alpha}} (1 - [m]_y)^{r+j} P(\theta_1 + [m]_y) P([m]_y + (1 - [m]_y)\theta_2) d\theta_1 d\theta_2 \ .
$$

Next note that $\sum_{n|m} n^{-1-i\alpha} = \frac{\phi(m)}{m} + O(|\alpha| \sum_{n|m} n^{-1})$. Thus we have

$$
c_{u,j} = \frac{(-1)^j j! \binom{r}{j} (\log y)^{2r+u+j}}{(r-1)!(r+j-1)!} \int_0^1 \int_0^1 \theta_1^{r-1} \theta_2^{r+j-1} (\eta^{-1} - \theta_1)^u \cdot
$$

$$
\sum_{m \le y^{1-\theta_1}} \frac{\phi(m)\sigma_r(m)^2}{m^2} (1 - [m]_y)^{r+j} P(\theta_1 + [m]_y) P([m]_y + (1 - [m]_y)\theta_2) d\theta_1 d\theta_2
$$

plus an error term of the shape

(6.19)

$$
\ll_{r,j,u} |\alpha| (\log y)^{2r+u+j} \sum_{m \le y} \frac{\sigma_r(m)^2}{m} \sum_{n|m} \frac{1}{n}
$$

$$
\ll_{r,j,u} |\alpha| L^{2r+u+j} \sum_{n \le y} \frac{\sigma_r(n)^2}{n^2} \sum_{k \le y/m} \frac{\sigma_r(k)^2}{k} \ll_{r,j,u} L^{r^2+2r+u+j-1} \ .
$$

Note that we can write down the constant in the O term explicitly in terms of r, j, and u. Applying Lemma 5.9 to the inner sum we derive

$$
c_{u,j} = \frac{a_{r+1}(-1)^j j! \binom{r}{j} (\log y)^{r^2+2r+u+j}}{(r-1)!(r+j-1)!(r^2-1)!}
$$

$$
\cdot \int_0^1 \int_0^1 \int_0^{1-\theta_1} \theta_1^{r-1} \theta_2^{r+j-1} (\eta^{-1} - \theta_1)^u \delta^{r^2-1} R(\delta) d\delta d\theta_1 d\theta_2
$$

where $R(\delta) = (1-\delta)^{r+j}P(\theta_1+\delta)P(\delta+(1-\delta)\theta_2)$ and this is valid up to an error of $O_{r,j,u}(L^{r^2+2r+u+j-1})$. If we recall the definition $Q_u(\delta) = \int_0^1 \theta_2^u P(\delta + (1-\delta)\theta_2)\,d\theta_2$ and then execute the integration in the θ_2-variable this becomes

$$c_{u,j} \sim \frac{a_{r+1}(-1)^j j!\binom{r}{j}(\log y)^{r^2+2r+u+j}}{(r-1)!(r+j-1)!(r^2-1)!}$$
$$\cdot \int_0^1 \int_0^{1-\theta_1} \theta_1^{r-1}(\eta^{-1}-\theta_1)^u \delta^{r^2-1}(1-\delta)^{r+j}P(\theta_1+\delta)Q_{r+j-1}(\delta)d\delta d\theta_1\ .$$

Recalling definitions (1.10) and (1.11) we have

$$(6.20) \quad c_{u,j} = \frac{(r-1)!C_r(-1)^j j!\binom{r}{j}(\log y)^{r^2+2r+u+j}}{(r+j-1)!}k_P(u,r+j,r+j-1)$$
$$+ O_{r,j,u}(L^{r^2+2r+u+j-1})\ .$$

Combining (6.16) and (6.20) establishes that I_{22} is $-(r-1)!C_r\frac{T}{2\pi}(\log y)^{r^2+2r+2}$ multiplied by the series

$$\sum_{j=2}^r \frac{(-1)^j\binom{r}{j}(i\alpha\log y)^{j-2}}{(r+j-1)!}\sum_{u=0}^\infty \frac{(i\alpha\log y)^u}{u!}k_P(u,r+j,r+j-1)$$
$$= \sum_{j=0}^{r-2} \frac{(-1)^j\binom{r}{j+2}(i\alpha\log y)^j}{(r+j+1)!}\sum_{u=0}^\infty \frac{(i\alpha\log y)^u}{u!}k_P(u,r+j+2,r+j+1)$$
$$= \sum_{j=0}^{r-2} \frac{(-1)^j\binom{r}{j+2}}{(r+j+1)!}\sum_{n=j}^\infty \frac{(\eta z)^n}{(n-j)!}k_P(n-j,r+j+2,r+j+1)$$

where in the second line we replaced $j-2$ by j and in the third line we made the variable change $n = u+j$ in the inner sum. Moreover, it can be checked that the error term $O_{r,j,u}(L^{r^2+2r+u+j-1})$ when substituted in (6.16) is smaller than the main term by a factor of $O(L^{-1})$. We now write $I_{22} = I_{22}' + I_{22}''$ where I_{22}' is the contribution from the $j=0$ term and I_{22}'' is the rest:

$$(6.21) \quad I_{22}' = -\frac{(r-1)C_r\frac{T}{2\pi}L^{(r+1)^2+1}}{2(r+1)}\sum_{n=0}^\infty \frac{z^n\eta^{n+(r+1)^2+1}}{n!}k_P(n,r+2,r+1)\ ,$$

(6.22)
$$I_{22}'' = -(r-1)!C_r\frac{T}{2\pi}L^{(r+1)^2+1}\sum_{n=1}^\infty z^n\eta^{n+(r+1)^2+1}$$
$$\cdot \sum_{1\le j\le\min(n,r-2)} \frac{(-1)^j\binom{r}{j+2}}{(n-j)!(r+j+1)!}k_P(n-j,r+j+2,r+j+1)\ .$$

6.4. **Evaluating I.** We collect our estimates to conclude the evaluation of I. Since $I = I_1+I_{21}+I_{22}'+I_{22}''$ plus error terms it follows from (6.13), (6.15), (6.21),

and (6.22) that
(6.23)
$$I \sim C_r \frac{T}{2\pi} L^{(r+1)^2+1} \left(\sum_{j=1}^{\infty} z^j \eta^{j+(r+1)^2+1} \left(\frac{r\hat{i}(r,\eta,j)}{j!} + \hat{k}_1(r,\eta,j) + \hat{k}_2(r,\eta,j) \right) \right)$$
$$+ \mathrm{CT}(I)$$

where $\mathrm{CT}(I)$ denotes the constant term in the above Taylor series,

$$\hat{i}(r,\eta,j) = -i_P(r,r,j,r-1,r-1)\eta^{-1} + (i_P(r+1,r,j,r,r-1) + i_P(r,r+1,j,r-1,r-1)),$$

$$\hat{k}_1(r,\eta,j) = -\frac{k_P(j+2,r,r-1)}{(j+2)!} + \frac{k_P(j+1,r+1,r)}{(j+1)!} - \frac{(r-1)k_P(j,r+2,r+1)}{2(r+1)j!},$$

$$\hat{k}_2(r,\eta,j) = -(r-1)! \sum_{u=1}^{\min(j,r-2)} \frac{(-1)^u \binom{r}{u+2}}{(j-u)!(r+u+1)!} k_P(j-u,r+u+2,r+u+1).$$

Next remark that we may conveniently combine $\hat{k}(r,\eta,j) = \hat{k}_1(r,\eta,j) + \hat{k}_2(r,\eta,j)$ to obtain
(6.24)

$$\hat{k}(r,\eta,j) = -(r-1)! \sum_{u=-2}^{\min(j,r-2)} \frac{(-1)^u \binom{r}{u+2}}{(j-u)!(r+u+1)!} k_P(j-u,r+u+2,r+u+1).$$

This completes the evaluation of I.

6.5. **The final details.** We now complete the proof of Theorem 2. In order to abbreviate the following equations we put

$$(6.25) \quad \theta = C_r \frac{T}{2\pi} L^{(r+1)^2+1}, \ a = \eta^{(r+1)^2-1}, \ b = \eta^{(r+1)^2}, \ \text{and} \ c = \eta^{(r+1)^2+2}.$$

Recall that the discrete moment we are evaluating satisfies

$$(6.26) \qquad m(H_r,T;\alpha) = 2\mathrm{Re}(I) - \overline{J} + O(yT^{1/2+\epsilon}).$$

Moreover, we showed (4.5) that $J = \mathrm{CT}(J)(1 + O(L^{-1}))$ where

$$\mathrm{CT}(J) = -\theta \left(a \int_0^1 \alpha^{r^2-1}(1-\alpha)^{2r} Q_{r-1}(\alpha)^2 \, d\alpha \right.$$
(6.27)
$$\left. -2b \int_0^1 \alpha^{r^2-1}(1-\alpha)^{2r+1} Q_{r-1}(\alpha) Q_r(\alpha) \, d\alpha \right).$$

We shall now combine (6.23) and (6.27) in (6.26) to finish the proof. In particular we shall now prove that $2\mathrm{CT}(I) = \mathrm{CT}(J)$ and hence $\mathrm{CT}(m(H_r,T,\alpha)) = 0$. This was expected since the constant term in the Taylor series of $\zeta(\rho+\alpha)$ is zero for each ρ. Moreover, this fact that the constant term must be zero provides a consistency check of our calculation. We now verify that $2\mathrm{CT}(I) = \mathrm{CT}(J)$. Recall that $\mathrm{CT}(I) = \mathrm{CT}(I_1) + \mathrm{CT}(I_{21}) + \mathrm{CT}(I'_{22})$. From (6.13) we have

$$\mathrm{CT}(I_1) =$$
$$r\theta \left(-b\, i_P(r,r,0,r-1,r-1) + c(i_P(r+1,r,0,r,r-1) + i_P(r,r+1,0,r-1,r)) \right).$$

Each of the above integrals has the form

$$(6.28) \qquad \int_0^1 \int_0^{1-x} x^{r^2-1}(1-x)^u(1-y-x)^v Q_{u-1}(x)Q_{v-1}(x+y)\, dy\, dx$$

for $(u,v) = (r,r), (r+1,r), (r,r+1)$. Note that we have the identity

$$(6.29) \qquad (1-x)^{n+1}Q_n(x) = \int_0^{1-x} \beta^n P(x+\beta)\, d\beta\ .$$

One may deduce from (6.29) that

$$\frac{1}{v}(1-x)^{v+1}Q_v(x) = \int_0^{1-x}(1-x-y)^v Q_{v-1}(x+y)\, dy$$

and hence

$$(6.28) = \frac{1}{v}\int_0^1 x^{r^2-1}(1-x)^{u+v+1}Q_{u-1}(x)Q_v(x)\, dx\ .$$

It follows that

$$\mathrm{CT}(I_1) = \theta \cdot \left(-b \int_0^1 x^{r^2-1}(1-x)^{2r+1}Q_{r-1}(x)Q_r(x)\, dx \right.$$

$$+c \left(\int_0^1 x^{r^2-1}(1-x)^{2r+2}Q_r(x)^2\, dx \right.$$

$$\left. \left. +\frac{r}{r+1}\int_0^1 x^{r^2-1}(1-x)^{2r+2}Q_{r-1}(x)Q_{r+1}(x)\, dx \right) \right)\ .$$

By (6.15) we have $\mathrm{CT}(I_{21}) = \theta\eta^{(r+1)^2+1}(k_P(1,r+1,r) - (1/2)k_P(2,r,r-1))$.
Expanding out the factor $(\eta^{-1} - \theta_1)^2$ in the definition of k_P we have

$$\theta\eta^{(r+1)^2+1}k_P(2,r,r-1) \sim \theta\eta^{(r+1)^2+1}$$

$$\cdot \sum_{j=0}^2 \binom{2}{j}(-1)^j \eta^{-(2-j)} \int_0^1\int_0^{1-\theta_1} \theta_1^{r-1+j}\delta^{r^2-1}(1-\delta)^r P(\delta+\theta_1)Q_{r-1}(\delta)\, d\delta\, d\theta_1\ .$$

However, by (6.29) this simplifies to

$$(6.30)$$

$$\theta\eta^{(r+1)^2+1}k_P(2,r,r-1) \sim \theta\left(a \int_0^1 \delta^{r^2-1}(1-\delta)^{2r}Q_{r-1}(\delta)^2\, d\delta \right.$$

$$-2b \int_0^1 \delta^{r^2-1}(1-\delta)^{2r+1}Q_{r-1}(\delta)Q_r(\delta)\, d\delta$$

$$\left. c \int_0^1 \delta^{r^2-1}(1-\delta)^{2r+2}Q_{r-1}(\delta)Q_{r+1}(\delta)\, d\delta \right)\ .$$

Moreover, a similar calculation establishes

$$(6.31)$$

$$\theta\eta^{(r+1)^2+1}k_p(1,r+1,r) \sim \theta$$

$$\left(b \int_0^1 \delta^{r^2-1}(1-\delta)^{2r+1}Q_{r-1}(\delta)Q_r(\delta)\, d\delta - c \int_0^1 \delta^{r^2-1}(1-\delta)^{2r+2}Q_r(\delta)^2\, d\delta \right)\ .$$

Combining (6.30) and (6.31) establishes

$$\mathrm{CT}(I_2) = \theta\left(-\frac{a}{2}\int_0^1 \delta^{r^2-1}(1-\delta)^{2r}Q_{r-1}(\delta)^2\,d\delta\right.$$

$$+2b\int_0^1 \delta^{r^2-1}(1-\delta)^{2r+1}Q_{r-1}(\delta)Q_r(\delta)\,d\delta$$

$$\left.-c\int_0^1 \delta^{r^2-1}(1-\delta)^{2r+2}\left(Q_r(\delta)^2 + \frac{1}{2}Q_{r-1}(\delta)Q_{r+1}(\delta)\right)d\delta\right).$$

In a similar way, it follows from (6.21)

$$\mathrm{CT}(I'_{22}) = -\theta\frac{(r-1)}{2(r+1)}c\int_0^1 \delta^{r^2-1}(1-\delta)^{2r+2}Q_{r-1}(\delta)Q_{r+1}(\delta)\,d\delta.$$

Combining constant terms yields $\mathrm{CT}(I) = \theta(c_1 a + c_2 b + c_3 c)$ where

$$c_1 = -\frac{1}{2}\int_0^1 \delta^{r^2-1}(1-\delta)^{2r}Q_{r-1}(\delta)^2\,d\delta,$$

$$c_2 = \int_0^1 \delta^{r^2-1}(1-\delta)^{2r+1}Q_{r-1}(\delta)Q_r(\delta)\,d\delta,$$

$$c_3 = \int_0^1 \delta^{r^2-1}(1-\delta)^{2r+2}$$

$$\cdot\left(Q_r(\delta)^2(1-1) + Q_{r-1}(\delta)Q_{r+1}(\delta)\left(\frac{r}{r+1} - \frac{1}{2} - \frac{(r-1)}{2(r+1)}\right)\right)d\delta.$$

Observe that $c_3 = 0$ and hence we have shown that

$$\mathrm{CT}(I) = \theta\cdot$$

$$\left(-\frac{a}{2}\int_0^1 \delta^{r^2-1}(1-\delta)^{2r}Q_{r-1}(\delta)^2\,d\delta + b\int_0^1 \delta^{r^2-1}(1-\delta)^{2r+1}Q_{r-1}(\delta)Q_r(\delta)\,d\delta\right).$$

However, glancing back at (6.27) we see that $2\mathrm{CT}(I) = \mathrm{CT}(J)$. By this fact, (6.23), (6.24), and (6.26) we finally deduce $m(H_r, T; \alpha)$ is asymptotic to

$$C_r\frac{T}{\pi}L^{(r+1)^2+1}\mathrm{Re}\left(\sum_{j=1}^{\infty}(i\alpha L)^j\eta^{j+(r+1)^2+1}\left(\frac{r\hat{i}(r,\eta,j)}{j!} + \hat{k}(r,\eta,j)\right)\right).$$

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