BOUNDS AND CONJECTURES FOR ADDITIVE DIVISOR SUMS

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In memory of Kevin Henriot.

ABSTRACT. Additive divisor sums play a prominent role in the theory of the moments of the Riemann zeta function. There is a long history of determining sharp asymptotic formula for the shifted convolution sum of the ordinary divisor function. In recent years, it has emerged that a sharp asymptotic formula for the shifted convolution sum of the triple divisor function would be useful in evaluating the sixth moment of the Riemann zeta function. In this article, a uniform lower bound of the correct order of magnitude is established for the shifted convolution sum of the k-th divisor function. In addition, the conjecture for the asymptotic formula for this additive divisor sum is studied. The leading term in the asymptotic formula is simplified and also a probabilistic method is presented which gives the same leading term.

1. INTRODUCTION AND MAIN THEOREM

Many important problems in analytic number theory concern sums of the form

$$\sum_{n \leq x} f(n)g(n+h) \tag{1.1}$$

where $h \in \mathbb{N}$ and f and g are arithmetic functions. For instance, the twin prime conjecture would follow from an asymptotic evaluation of (1.1) with $f = g = \Lambda$, the von Mangoldt function. If $f = g = \lambda$, the Liouville function, this is a special case of the sum that occurs in Chowla's conjecture. In this article, we focus on (1.1) with f and g equal to τ_k, the k-th divisor function where $k \in \mathbb{N}$. For $n \in \mathbb{N}$ it is defined by

$$\tau_k(n) = \#\{(n_1, \ldots, n_k) \in \mathbb{N}^k \mid n_1 \cdots n_k = n\}.$$

Equivalently, $\tau_k(n)$ is the n-th coefficient of the Dirichlet series of $\zeta(s)^k$, where $\zeta(s)$ is the Riemann zeta function. Our main focus is the correlation sum

$$D_k(x, h) := \sum_{n \leq x} \tau_k(n)\tau_k(n+h) \text{ with } h \in \mathbb{N}. \tag{1.2}$$

For $k = 1$, this sum is trivial. For $k = 2$, there is a rich theory connecting this sum to the spectral theory of automorphic forms. However, for $k > 2$, this sum is mysterious and there are few results. Nevertheless, there is the following conjecture:

Additive Divisor Conjecture (simplified version). Let $\varepsilon > 0$ and $k \geq 2$. For $1 \leq h \leq x^{1-\varepsilon}$, we have

$$D_k(x, h) := \sum_{n \leq x} \tau_k(n)\tau_k(n+h) \sim \frac{C_k f_k(h)}{(k-1)!^2} x(\log x)^{2k-2}$$

as $x \to \infty$, where

$$(1.3) \qquad C_k := \prod_p \left(2\left(1 - \frac{1}{p}\right)^{k-1} - \left(1 - \frac{1}{p}\right)^{2k-2} \right),$$

$f_k(h)$ is a multiplicative function defined on prime powers p^α by

$$(1.4)$$
$$f_k(p^\alpha) := \sum_{j=0}^{\alpha} \left(\frac{\tau_{k-1}(p^j)^2 H_{k-1,j}(p^{-1})^2}{p^j} - \frac{\tau_{k-1}(p^{j+1})^2 H_{k-1,j+1}(p^{-1})^2}{p^{j+2}} \right) \left(2\left(1 - \frac{1}{p}\right)^{k-1} - \left(1 - \frac{1}{p}\right)^{2k-2} \right)^{-1},$$

and

$$(1.5) \qquad H_{\ell,j}(x) := \sum_{i=0}^{\ell-1} \frac{\binom{\ell-1}{i}}{\binom{j+i}{j}} (1-x)^{\ell-1-i} x^i \quad \text{where } \ell \in \mathbb{N}, j \in \mathbb{Z}_{\geq 0}$$

is a polynomial of degree $\ell - 1$.

This is a simplified version of conjectures of Ivic [28] and Conrey-Gonek [9], though in the above formulation we allow h to be as large as $x^{1-\varepsilon}$ instead of $x^{\frac{1}{2}}$. It is unclear whether (1.4) is the simplest or even the most natural expression for $f_k(p^\alpha)$. Several other expressions are given in (1.19) and (4.11) below. The case $h = 1$ reduces to

$$\sum_{n \leq x} \tau_k(n) \tau_k(n+1) \sim \frac{C_k}{(k-1)!^2} x (\log x)^{2k-2}.$$

The conjectures of [28] and [9] may be written in the form

$$(1.6) \qquad D_k(x, h) = x \left(\alpha_0(h)(\log x)^{2k-2} + \sum_{i=1}^{2k-2} \alpha_i(h)(\log x)^{2k-2-i} \right) + o(x)$$

for certain coefficients $\alpha_i(h)$ where h is allowed to vary with x. Ivic [28] gave formulae for the $\alpha_i(h)$ in terms of certain singular series. On the other hand, Conrey and Gonek gave a formula for the derivative of the above main term in terms of a complicated double complex integral. This will be discussed in further detail in section two where we show that $\alpha_0(h) = C_k f_k(h)/(k-1)!^2$.

The main result in this article is a uniform lower bound for $D_k(x, h)$.

Theorem 1.1. *For $k \geq 3$, there exists $B_k > 0$ such that for $h \leq \exp(B_k (\log x \log \log x)^{\frac{k-1}{k-1.99}})$, we have*

$$\frac{1}{2^{2k-2}} \frac{C_k f_k(h)}{(k-1)!^2} x (\log x)^{2k-2} \left(1 + O_k\left(\frac{\log \log h}{\log x} \right) \right) \leq D_k(x, h)$$

as $x \to \infty$.

This provides some evidence towards the Additive Divisor Conjecture. In an earlier version of this article we also proved that

$$D_k(x, h) \ll_k \prod_{p|h} \left(1 + \frac{k(k-1)}{p} \right) x (\log x)^{2k-2}.$$

This was deduced from a Brun-Titchmarsh type bound for $\tau_k(n)$ as proven in [32] and [42, Theorem 2]. However, Kevin Henriot recently informed us that S. Daniel [11] showed that

$$(1.7) \qquad D_k(x, h) \ll_k \prod_{p|h} \left(1 + \frac{(k-1)^2}{p} \right) x (\log x)^{2k-2},$$

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for $h \le x^C$, for any $C > 0$. Note that since

(1.8)
$$f_k(p^\alpha) = 1 + \frac{(k-1)^2}{p} + O_k(p^{-2})$$

(1.7) implies

(1.9)
$$D_k(x, h) \ll_k f_k(h)x(\log x)^{2k-2},$$

for $h \le x^C$, for any $C > 0$. Unfortunately, this result was never published. However, Henriot has shown us a proof [25] based on [23] and [24]. In [23] he establishes bounds for

(1.10)
$$\sum_{x < n \le x+y} \tau_{k_1}(|Q_1(n)|)\tau_{k_2}(|Q_2(n)|)\cdots\tau_{k_J}(|Q_J(n)|)$$

where Q_j are polynomials with integer coefficients. More generally he bounds

(1.11)
$$\sum_{x < n \le x+y} f_1(|Q_1(n)|)f_2(|Q_2(n)|)\cdots f_J(|Q_J(n)|)$$

where the f_i belong to a general class of multiplicative functions. Such expressions were originally considered by Nair and Tenenbaum [36]. However, their bounds for (1.11) were not uniform in the coefficients of the Q_j. This problem was addressed by Daniel [11] and Henriot [11]. Recently Klurman [31] has obtained some interesting results for (1.11) in the case that the images of the multiplicative functions f_i lie in the unit disc. Theorem 1.1 and (1.9) lead us to propose the following problem.

Problem. Let $k \ge 3$. Determine the best explicit constants a_k and b_k such that

$$a_k \le \frac{D_k(x, h)}{\frac{C_k f_k(h)}{(k-1)!^2}x(\log x)^{2k-2}} \le b_k,$$

uniformly for $h \le x^{1-\varepsilon}$, as $x \to \infty$.

Theorem 1 yields $a_k = \frac{1}{2^{2k-2}} - \varepsilon$. and (1.9) yields $b_k = O_k(1)$. Henriot has suggested that the proof of (1.7) demonstrates that b_k is doubly or triply exponential in k.

Another key goal of this article is to simplify the leading term in the conjectural formula for $D_k(x, h)$. Namely, we show that the argument using the δ-method suggests that $D_k(x, h) \sim \frac{f_k(h)}{(k-1)!^2}x(\log x)^{2k-2}$. In addition, we present an alternate probabilistic argument which also leads to this conjectural asymptotic. To finish this section, we give some properties of divisor functions, list our conventions and notation, and provide an outline of the article.

1.1. Properties of Divisor functions.

This article makes extensive use of divisor functions and related arithmetic functions. Recall that for $\ell \in \mathbb{N}$, the ℓ-th divisor function satisfies

(1.12)
$$\sum_{j=0}^{\infty} \tau_\ell(p^j)X^j = (1 - X)^{-\ell}$$

for p prime and $|X| < 1$. It follows that for p prime and $j \ge 0$,

$$\tau_\ell(p^j) = \binom{\ell + j - 1}{j}.$$

The divisor functions satisfy the relation

(1.13)
$$\tau_{\ell-1}(p^j) = \tau_\ell(p^j) - \tau_\ell(p^{j-1}) \text{ for } p \text{ prime}, \ell, j \ge 1.$$

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We shall also encounter a multiplicative function $\sigma_\ell(\cdot, s) : \mathbb{N} \to \mathbb{C}$, where $\ell \in \mathbb{N}$, $s \in \mathbb{C}$. For $n \in \mathbb{N}$, it is defined by

$$(1.14) \qquad \sigma_\ell(n, s) = \Big(\sum_{a=1}^{\infty} \frac{\tau_\ell(na)}{a^s} \Big) \zeta(s)^{-\ell}.$$

By multiplicativity, it follows that

$$(1.15) \qquad \sigma_\ell(p^j, s) = \frac{\sum_{i=0}^{\infty} \frac{\tau_\ell(p^{j+i})}{p^{is}}}{\sum_{i=0}^{\infty} \frac{\tau_\ell(p^i)}{p^{is}}} = (1 - p^{-s})^\ell \sum_{i=0}^{\infty} \frac{\tau_\ell(p^{j+i})}{p^{is}}$$

for $j \geq 1$, and in particular,

$$(1.16) \qquad \frac{\sigma_\ell(p, s)}{p^s} = 1 - (1 - p^{-s})^\ell.$$

Moreover, it was proven in [37] that

$$(1.17) \qquad \sigma_\ell(p^j, s) = \tau_\ell(p^j) H_{\ell, j}(p^{-s})$$

where

$$(1.18) \qquad H_{\ell, j}(x) := jx^{-j} \int_0^x t^{j-1}(1 - t)^{\ell - 1} \, dt \quad [1]$$

for $j, \ell \in \mathbb{N}$. Note that $H_{\ell, j}(x)$ is a degree $\ell - 1$ polynomial and $H_{\ell, j}(0) = 1$. Repeated integration by parts of (1.18) leads to the representation given by (1.5). Note that (1.4) and (1.17) imply

$$(1.19) \qquad f_k(p^\alpha) := \sum_{j=0}^{\alpha} \Big(\frac{\sigma_{k-1}(p^j, 1)^2}{p^j} - \frac{\sigma_{k-1}(p^{j+1}, 1)^2}{p^{j+2}} \Big) \Big(2\Big(1 - \frac{1}{p}\Big)^{k-1} - \Big(1 - \frac{1}{p}\Big)^{2k-2} \Big)^{-1}.$$

At several points in this article we make use of this representation instead of (1.4).

1.2. **Conventions and notation.** In this article we shall use the convention that ε denotes an arbitrarily small positive constant which may vary from line to line. Given two functions $f(x)$ and $g(x)$, we shall interchangeably use the notation $f(x) = O(g(x))$, $f(x) \ll g(x)$, and $g(x) \gg f(x)$ to mean there exists $M > 0$ such that $|f(x)| \leq M|g(x)|$ for sufficiently large x. If we write $f(x) = O_k(g(x))$, $f(x) \ll_k g(x)$, or $f(x) \asymp_k g(x)$, then we mean that the corresponding constants depend on k. The letter p will always be used to denote a prime number. For a complex valued, differentiable function $F : \mathbb{C}^2 \to \mathbb{C}$ and $i_1, i_2 \in \mathbb{Z}_{\geq 0}$ we write

$$(1.20) \qquad F^{(i_1, i_2)}(s_1, s_2) := \frac{\partial^{i_1}}{\partial s_1^{i_1}} \frac{\partial^{i_2}}{\partial s_2^{i_2}} F(s_1, s_2)$$

where $\frac{\partial^i}{\partial s^i}$ denotes the i-th partial derivative with respect to s.

Given $a, b \in \mathbb{Z}$, we let (a, b) denote the greatest common divisor of a and b and $[a, b]$ denotes the least common multiple of a and b.

1.3. **Organization of the article.** The article is organized as follows. In section 2 the conjectural asymptotic formula for $D_k(x, h)$ is studied based on the work of Ivic [28] and Conrey-Gonek [9]. We show that leading term in the asymptotic formula for $D_k(x, h)$ is $\frac{1}{(k-1)!^2} C_k f_k(h) x (\log x)^{2k-2}$. In section 3, the lower bound in Theorem 1.1 is proven. In section 4, a simple probabilistic method is used to rederive the main

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[1]In [37], we used the notation $H_{j, \ell}(x)$ instead of $H_{\ell, j}(x)$.

term of $D_k(x,h)$ which agrees with the calculation in section 2. Finally, we discuss open problems related to additive divisor sums and avenues for future research.

2. A brief history of additive divisor sums and a conjectural formula for $D_k(x,h)$

2.1. **A history of additive divisor sums.** Questions concerning sums of the form $D_k(x,h)$ are called additive divisor problems. These functions are of interest due to their connection to the $2k$-th moments of the Riemann zeta function, defined by

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt \text{ for } k \geq 0.$$

In 1926, Ingham [26] discovered that $D_2(x,h)$ is intimately related to the fourth moment, $I_2(T)$. He succeeded in proving that

$$I_2(T) \sim \frac{T}{2\pi^2}(\log T)^4$$

and an important part of his argument made use of the inequality

$$D_2(x,h) \ll \sigma_{-1}(h)x(\log x)^2$$

for $h \leq x$, where $\sigma_{-1}(h) = \sum_{d|h} d^{-1}$. In [27] he improved this to

$$(2.1) \qquad D_2(x,h) \sim \frac{6}{\pi^2}\sigma_{-1}(h)x \log^2 x.$$

In 1931, Estermann [17] proved an estimate of the shape

$$(2.2) \qquad D_2(x,h) = x\left(\frac{6}{\pi^2}\sigma_{-1}(h)\log^2 x + \alpha_1(h)\log x + \alpha_2(h)\right) + O(x^{\theta+\varepsilon})$$

with $\theta = \frac{11}{12}$ and $\alpha_1(h)$ and $\alpha_2(h)$ are certain arithmetic functions. Estermann's work relates $D_2(x,h)$ to a formula involving special exponential sums known as Kloosterman sums. For q a natural number and u, v integers, the Kloosterman sum $S(u,v;q)$ is defined by

$$S(u,v;q) := \sum_{\substack{a=1 \\ (a,q)=1 \\ a\bar{a}\equiv 1 (\mathrm{mod}\ q)}}^{q} e\left(\frac{ua + v\bar{a}}{q}\right).$$

These sums exhibit considerable cancellation and they arise in many contexts in analytic number theory. Estermann derived the non-trivial bound $S(u,v;q) \ll q^{\frac{3}{4}+\varepsilon}(u,q)^{\frac{1}{4}}$ and this led to the error term in (2.2). A famous result due to Weil is the bound: $|S(u,v;q)| \leq \tau_2(q)(q,u,v)^{1/2}q^{1/2}\tau(q)$. Much later, Heath-Brown [22] made use of Weil's bound to obtain (2.2) with $\theta = \frac{5}{6}$. From this he deduced that there exists a degree four polynomial Q_4 such that

$$(2.3) \qquad I_2(T) = TQ_4(\log T) + O(T^{\Theta+\varepsilon}),$$

where $\Theta = \frac{5}{6}$ is valid. The next advance was due to Deshouillers and Iwaniec [13], who proved that (2.2) is valid with $\theta = \frac{2}{3}$, in the case $h = 1$. In their work, they related $D_2(x;1)$ to averages of Kloosterman sums and then made use of Kuznetsov's formula. This is a formula which relates sums of Kloosterman sums to the coefficients of Maass wave forms and holomorphic modular forms. Motohashi extended this method and obtained (2.2) with $\theta = \frac{2}{3}$, uniformly for $h \leq x^{\frac{20}{27}}$. He proved

$$(2.4) \qquad D_2(x,h) = \frac{6}{\pi^2}\int_0^{\frac{x}{h}} q_2(t,h)\ dt + E_2(x,h)$$

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where

$$
q_2(t,h) = \sigma(h)\log(t)\log(t+1) + (\sigma(h)(2\gamma - \frac{\zeta'}{\zeta}(2) - \log(h)) + 2\sigma^{(1)}(h))\log(t(t+1))
$$

(2.5)

$$
+ \sigma(h)\Big((2\gamma - 2\frac{\zeta'}{\zeta}(2) - \log h)^2 - 4\Big(\frac{\zeta'}{\zeta}\Big)'(2)\Big) + 4\sigma^{(1)}(h)(2\gamma - 2\frac{\zeta'}{\zeta}(2) - \log h) + 4\sigma^{(2)}(h),
$$

$\sigma^{(j)}(h) := \sum_{d|h} d(\log d)^j$, and γ is Euler's constant and

(2.6)
$$
E_2(x,h) = O((x(x+h))^{\frac{1}{3}+\varepsilon} + h^{\frac{9}{40}}(x(x+h))^{\frac{1}{4}+\varepsilon} + h^{\frac{7}{10}}x^\varepsilon).
$$

Related work of Motohashi establishes that $\Theta = \frac{2}{3}$ is valid in (2.3). Meurman [33] showed that

(2.7)
$$
E_2(x,h) = O((x(x+h))^{\frac{1}{3}+\varepsilon} + (x(x+h))^{\frac{1}{4}}x^\varepsilon \min(x^{\frac{1}{4}}, h^{\frac{1}{8}+\frac{\alpha}{2}})),
$$

where α is a positive constant which satisfies

(2.8)
$$
|\rho_j(n)| \le n^\alpha |\rho_j(1)|
$$

where $\{\rho_j(n)\}_{n=1}^\infty$ are the Fourier coefficients of an orthonormal basis of the space of non-holomorphic cusp forms for the full modular group.

Some closely related classes of divisor sums to $D_2(x;h)$ have been studied with some success. Let $k, \ell \in \mathbb{N}$ and set

$$
D_{k,\ell}(x,h) := \sum_{n \le x} \tau_k(n)\tau_\ell(n+h).
$$

Observe that $D_k(x,h) = D_{k,k}(x,h)$. Linnik developed highly original techniques using ideas from additive number theory and probability theory, most notably the dispersion method [32] to deal with $D_{k,2}(x,h)$ with $k \ge 2$. He proved an asymptotic formula for $D_{k,2}(x,h)$, obtaining the leading term with an error term. The error term was improved by Motohashi [34]., who used large sieve methods. Recently, Topacogullari [46] established a main term with a power savings in the case of $D_{3,2}(x,h)$. This filled in details of results, stated without proof, by Deshouillers [12] and Bykovski and Vinogradov [6]. Furthermore, Drappeau [14] has recently provided a main term with a power savings in the error term for $D_{k,2}(x,h)$ with $k \ge 3$ and this too has recently been improved by Topacogullari [48]. Despite these impressive results, no asymptotic formula for $D_{k,\ell}(x,h)$ has been proven in the case both k and ℓ are greater than two. We now present a conjectural formula for $D_k(x,h)$.

2.2. **A conjectural formula for $D_k(x,h)$.** We follow the work of Ivic and Conrey and Gonek to work out the leading term of the conjectured main term for $D_k(x,h)$. We shall be concerned with an expression of the form $D_k(x,h) = m_k(x,h) + E_k(x,h)$ where $m_k(x,h)$ is the "main term" and $E_k(x,h)$ is the "error term." In [28], [29], and [9], $m_k(x,h)$ was studied via Duke, Friedlander, and Iwaniec's [16] version of the circle method, known as the δ-method. One of the key ideas of the circle method is to detect an additive condition via additive characters. Consequently, it is important to have an asymptotic formula for the exponential sums $\sum_{n \le x} \tau_k(n)e(\frac{an}{q})$ where $(a,q) = 1$ and $e(\theta) := e^{2\pi i \theta}$. Naturally, one must understand the Dirichlet series $\sum_{n=1}^\infty \tau_k(n)e(\frac{an}{q})n^{-s}$. Ivic [29] obtained a meromorphic continuation of this series by decomposing it in terms of Hurwitz zeta functions. On the other hand, Conrey-Gonek [9] obtained a meromorphic continuation by expressing $e(\frac{an}{q})$ in terms of multiplicative Dirichlet characters. They showed that

(2.9)
$$
\sum_{n \le x} \tau_k(n)e\Big(\frac{an}{q}\Big) \sim \frac{1}{q}\int_0^x P_k(t,q)dt
$$

where $P_k(t, q)$ is defined by

$$(2.10) \qquad P_k(t, q) = \frac{1}{2\pi i} \int_C \zeta(s+1)^k G_k(q, s+1) \left(\frac{x}{q}\right)^s ds,$$

$C = \{z \in \mathbb{C} \mid |z| = \eta\}$ for $0 < \eta < \frac{1}{10}$, and for $k \in \mathbb{N}$, $s \in \mathbb{C}$, $G_k(\cdot, s) : \mathbb{N} \to \mathbb{C}$ is the multiplicative function defined by [2]

$$(2.11) \qquad G_k(n, s) = \sum_{a|n} \frac{\mu(a)a^s}{\phi(a)} \sum_{b|a} \frac{\mu(b)}{b^s} \sigma_k\left(\frac{nb}{a}, s\right).$$

Using (2.9), the δ-method leads to

$$(2.12) \qquad m_k(x, h) = \int_0^x \sum_{q=1}^{\infty} \frac{c_q(h)}{q^2} P_k(t, q) P_k(t+h, q) dt$$

where $c_q(h) = \sum_{\substack{a=1 \\ (a,q)=1}}^q e(\frac{ah}{q})$ is the Ramanujan sum. From the identity $\log(t+h) = \log t + O(h/t)$ (see [28]), it follows that

$$(2.13) \qquad m_k(x, h) = \int_0^x \sum_{q=1}^{\infty} \frac{c_q(h)}{q^2} P_k(t, q)^2 dt + O(ht^\varepsilon).$$

We now simplify the integrands in (2.12) and (2.13). We denote them as

$$(2.14) \qquad q_k(t, h) := \sum_{q=1}^{\infty} \frac{c_q(h)}{q^2} P_k(t, q) P_k(t+h, q)$$

and

$$(2.15) \qquad r_k(t, h) := \sum_{q=1}^{\infty} \frac{c_q(h)}{q^2} P_k(t, q)^2.$$

Observe that (2.13) implies

$$(2.16) \qquad \int_0^x q_k(t, h) dt = \int_0^x r_k(t, h) + O(ht^\varepsilon).$$

We first calculate $q_k(t, h)$. Applying (2.10) twice, it follows that

$$(2.17) \qquad q_k(t, h) = \frac{1}{(2\pi i)^2} \int_{C_2} \int_{C_1} \zeta^k(s_1+1) \zeta^k(s_2+1) \mathcal{D}_k(s_1, s_2) t^{s_1}(t+h)^{s_2} ds_1 ds_2,$$

$C_1 = \{s_1 \in \mathbb{C} \mid |s_1| = r_1\}$, $C_2 = \{s_2 \in \mathbb{C} \mid |s_2| = r_2\}$, $0 < r_1, r_2 < \frac{1}{10}$, and

$$(2.18) \qquad \mathcal{D}_k(s_1, s_2) = \sum_{q=1}^{\infty} \frac{c_q(h) G_k(q, s_1+1) G_k(q, s_2+1)}{q^{2+s_1+s_2}}.$$

We now apply the residue theorem to the inner integral in (2.17). For each $k \in \mathbb{N}$, there exist constants $\alpha_{j,k}$ with $j \geq 0$ such that

$$(2.19) \qquad \zeta^k(s_1+1) = s_1^{-k}(\alpha_{0,k} + \alpha_{1,k}s_1 + \alpha_{2,k}s_1^2 + \cdots), \quad \text{where } \alpha_{0,k} = 1.$$

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[2]Conrey and Gonek use the notation $G_k(s, n)$, whereas we use $G_k(n, s)$.

Furthermore, since

(2.20) $$\mathcal{D}_k(s_1, s_2) = \mathcal{D}_k^{(0,0)}(0, s_2) + \mathcal{D}_k^{(1,0)}(0, s_2)s_1 + \frac{1}{2}\mathcal{D}_k^{(2,0)}(0, s_2)s_1^2 + \cdots, \text{ and}$$

(2.21) $$t^{s_1} = 1 + (\log t)s_1 + \frac{1}{2}(\log t)^2 s_1^2 + \cdots$$

it follows that

$$\frac{1}{2\pi i}\int_{C_1} \zeta^k(s_1 + 1)\mathcal{D}_k(s_1, s_2)t^{s_1}ds_1 = \sum_{\substack{i_1+i_2+i_3=k-1 \\ i_1,i_2,i_3 \geq 0}} \frac{\alpha_{i_1,k}\mathcal{D}_k^{(i_2,0)}(0, s_2)(\log t)^{i_3}}{i_2!i_3!}.$$

Thus

$$q_k(t, h) = \sum_{\substack{i_1+i_2+i_3=k-1 \\ i_1,i_2,i_3 \geq 0}} \frac{\alpha_{i_1,k}(\log t)^{i_3}}{i_2!i_3!}\frac{1}{2\pi i}\int_{C_2} \zeta^k(s_2 + 1)\mathcal{D}_k^{(i_2,0)}(0, s_2)(t + h)^{s_2}ds_2.$$

For each value of i_2, a similar calculation establishes

$$\frac{1}{2\pi i}\int_{C_1} \zeta^k(s_2 + 1)\mathcal{D}_k^{(i_2,0)}(0, s_2)(t + h)^{s_2}ds_2 = \sum_{\substack{j_1+j_2+j_3=k-1 \\ j_1,j_2,j_3 \geq 0}} \frac{\alpha_{j_1,k}\mathcal{D}_k^{(i_2,j_2)}(0, 0)(\log(t + h))^{j_3}}{j_2!j_3!}$$

and hence

(2.22) $$q_k(t, h) = \sum_{\substack{i_1,i_2,i_3 \geq 0 \\ i_1+i_2+i_3=k-1}} \frac{\alpha_{i_1,k}(\log t)^{i_3}}{i_2!i_3!} \sum_{\substack{j_1,j_2,j_3 \geq 0 \\ j_1+j_2+j_3=k-1}} \frac{\alpha_{j_1,k}\mathcal{D}_k^{(i_2,j_2)}(0, 0)(\log(t + h))^{j_3}}{j_2!j_3!}.$$

An analogous computation establishes

(2.23) $$r_k(t, h) = \sum_{\substack{i_1,i_2,i_3 \geq 0 \\ i_1+i_2+i_3=k-1}} \frac{\alpha_{i_1,k}(\log t)^{i_3}}{i_2!i_3!} \sum_{\substack{j_1,j_2,j_3 \geq 0 \\ j_1+j_2+j_3=k-1}} \frac{\alpha_{j_1,k}\mathcal{D}_k^{(j_2,i_2)}(0, 0)(\log t)^{j_3}}{j_2!j_3!}.$$

Formally, (2.23) is obtained from (2.22) by replacing each $\log(t + h)$ by $\log(t)$. Observe that (2.23) can be further simplified. Let $i = i_3 + j_3$ and note that $0 \leq i \leq 2k - 2$ so that

(2.24) $$r_k(t, h) = \sum_{i=0}^{2k-2} \alpha_i(h)(\log x)^{2k-2-i}$$

where $\alpha_0(h) = \frac{\mathcal{D}_k(0,0)}{(k-1)!^2}$, the term arising from $(i_1, i_2, i_3) = (j_1, j_2, j_3) = (0, 0, k - 1)$. We now show that $\mathcal{D}_k(0, 0) = C_k f_k(h)$. This will be deduced from the following lemma. This lemma will also be used in our proof of Theorem 1.1.

Lemma 2.1. *Let f_1, f_2 be nonzero multiplicative functions, τ_1, τ_2 be real numbers, and h a natural number such that*

(2.25) $$\mathcal{S}(\tau_1, \tau_2; h) := \sum_{g|h}\frac{1}{g^{\tau_1}}\sum_{d=1}^{\infty}\frac{\mu(d)f_1(gd)f_2(gd)}{d^{\tau_2}}$$

is absolutely convergent.
(i) We have

(2.26) $$\mathcal{S}(\tau_1, \tau_2; h) := \sum_{g|h}\frac{1}{g^{\tau_1}}\prod_{(p,g)=1}\left(1 - \frac{f_1(p)f_2(p)}{p^{\tau_2}}\right)\prod_{p^{\alpha}||g}\left(f_1(p^{\alpha})f_2(p^{\alpha}) - \frac{f_1(p^{\alpha+1})f_2(p^{\alpha+1})}{p^{\tau_2}}\right).$$

(ii) If for every prime p, $f_1(p)f_2(p) \neq p^{\tau_2}$, then

$$(2.27) \quad \mathcal{S}(\tau_1, \tau_2; h) = \prod_p \left(1 - \frac{f_1(p)f_2(p)}{p^{\tau_2}}\right) \prod_{p^\alpha || h} \sum_{j=0}^{\alpha} \left(\frac{f_1(p^j)f_2(p^j)}{p^{\tau_1 j}} - \frac{f_1(p^{j+1})f_2(p^{j+1})}{p^{\tau_1 j + \tau_2}}\right)\left(1 - \frac{f_1(p)f_2(p)}{p^{\tau_2}}\right)^{-1}.$$

Proof of Lemma 2.1. For each $g \mid h$, write $g = \prod_{p^\alpha || g} p^\alpha$. By multiplicativity of the inner summand it follows that

$$\mathcal{S}(\tau_1, \tau_2; h) := \sum_{g|h} \frac{1}{g^{\tau_1}} \left(\prod_{(p,g)=1} \sum_{m=0}^{\infty} \frac{\mu(p^m) f_1(p^m) f_2(p^m)}{(p^m)^{\tau_2}}\right) \cdot \left(\prod_{p^\alpha || g} \sum_{m=0}^{\infty} \frac{\mu(p^m) f_1(p^{m+\alpha}) f_2(p^{m+\alpha})}{(p^m)^{\tau_2}}\right).$$

Simplifying this expression, using that $\mu(1) = 1, \mu(p) = -1$, and $\mu(p^m) = 0$ for $m \geq 2$,

$$\mathcal{S}(\tau_1, \tau_2; h) := \sum_{g|h} \frac{1}{g^{\tau_1}} \prod_{(p,g)=1} \left(1 - \frac{f_1(p)f_2(p)}{p^{\tau_2}}\right) \prod_{p^\alpha || g} \left(f_1(p^\alpha)f_2(p^\alpha) - \frac{f_1(p^{\alpha+1})f_2(p^{\alpha+1})}{p^{\tau_2}}\right)$$

Since $f_1(p)f_2(p) \neq p^{\tau_2}$, we multiply and divide each summand by $\prod_{p|g}\left(1 - \frac{f_1(p)f_2(p)}{p^{\tau_2}}\right)$ to obtain

$$\mathcal{S}(\tau_1, \tau_2; h) := \sum_{g|h} \frac{1}{g^{\tau_1}} \prod_p \left(1 - \frac{f_1(p)f_2(p)}{p^{\tau_2}}\right) \prod_{p^\alpha || g} \left(f_1(p^\alpha)f_2(p^\alpha) - \frac{f_1(p^{\alpha+1})f_2(p^{\alpha+1})}{p^{\tau_2}}\right)\left(1 - \frac{f_1(p)f_2(p)}{p^{\tau_2}}\right)^{-1}$$

$$(2.28) \qquad = \prod_p \left(1 - \frac{f_1(p)f_2(p)}{p^{\tau_2}}\right) \sum_{g|h} \frac{1}{g^{\tau_1}} \prod_{p^\alpha || g} \left(f_1(p^\alpha)f_2(p^\alpha) - \frac{f_1(p^{\alpha+1})f_2(p^{\alpha+1})}{p^{\tau_2}}\right)\left(1 - \frac{f_1(p)f_2(p)}{p^{\tau_2}}\right)^{-1}.$$

Let r be a multiplicative function defined on prime powers by

$$r(p^\alpha) = \left(f_1(p^\alpha)f_2(p^\alpha) - \frac{f_1(p^{\alpha+1})f_2(p^{\alpha+1})}{p^{\tau_2}}\right)\left(1 - \frac{f_1(p)f_2(p)}{p^{\tau_2}}\right)^{-1}.$$

The sum in (2.28) equals $\sum_{g|h} r(g)g^{-\tau_1}$. By multiplicativity,

$$\sum_{g|h} r(g)g^{-\tau_1} = \prod_{p^\alpha || h} \sum_{j=0}^{\alpha} r(p^j)p^{-j\tau_1} = \prod_{p^\alpha || h} \sum_{j=0}^{\alpha} \left(\frac{f_1(p^j)f_2(p^j)}{p^{\tau_1 j}} - \frac{f_1(p^{j+1})f_2(p^{j+1})}{p^{\tau_1 j + \tau_2}}\right)\left(1 - \frac{f_1(p)f_2(p)}{p^{\tau_2}}\right)^{-1}.$$

Inserting this expression in (2.28) we derive (2.27). $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Using the above, we now demonstrate

$$(2.29) \qquad\qquad\qquad\qquad\qquad \mathcal{D}_k(0,0) = C_k f_k(h).$$

Inserting the identity $c_q(h) = \sum_{d|h, d|q} d\mu(q/d)$ in (2.18), exchanging summation, and making the variable change $q \to qd$ leads to

$$\mathcal{D}_k(s_1, s_2) = \sum_{d|h} \frac{1}{d^{1+s_1+s_2}} \sum_{q=1}^{\infty} \frac{\mu(q) G_k(qd, s_1+1) G_k(qd, s_2+1)}{q^{2+s_1+s_2}}.$$

This is now in the form of the previous lemma. We set $s_1 = s_2 = 0$, $f_1(n) = G_k(n,1)$, $f_2(n) = G_k(n,1)$, $\tau_1 = 1$, and $\tau_2 = 2$, to obtain

$$(2.30) \qquad\qquad\qquad\qquad \mathcal{D}_k(0,0) = \prod_p \mathcal{E}_k(p,0) \prod_{p^\alpha || h} \sum_{j=0}^{\alpha} \mathcal{E}_k(p,j) \mathcal{E}_k(p,0)^{-1},$$

where

$$(2.31) \qquad\qquad\qquad\qquad\qquad \mathcal{E}_k(p,j) := \frac{G_k(p^j,1)^2}{p^j} - \frac{G_k(p^{j+1},1)^2}{p^{j+2}}.$$

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We now show that $G_k(p^j, 1) = \sigma_{k-1}(p^j, 1)$. Observe that by [9, p. 592]

$$G_k(p^j, 1) = \left(1 - \frac{1}{p}\right)^{-1} (\sigma_k(p^j, 1) - \sigma_k(p^{j-1}, 1)). \tag{2.32}$$

Thus by (1.15),

$$G_k(p^j, 1) = \left(1 - \frac{1}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^k \sum_{i=0}^{\infty} \frac{\tau_k(p^{j+i}) - \tau_k(p^{j-1+i})}{p^{is}} = \left(1 - \frac{1}{p}\right)^{k-1} \sum_{i=0}^{\infty} \frac{\tau_{k-1}(p^{j+i})}{p^{is}} = \sigma_{k-1}(p^j, 1),$$

by definition. Hence,

$$\mathcal{E}_k(p, j) := \frac{\sigma_{k-1}(p^j, 1)^2}{p^j} - \frac{\sigma_{k-1}(p^{j+1}, 1)^2}{p^{j+2}}. \tag{2.33}$$

Observe that

$$\mathcal{E}_k(p, 0) = 1 - \frac{\sigma_{k-1}(p, 1)^2}{p^2} = 1 - (1 - (1 - p^{-1})^k)^2 = 2(1 - p^{-1})^k - (1 - p^{-1})^{2k}. \tag{2.34}$$

Thus, by (2.34), the first factor in (2.30) is C_k and by (2.33) the second factor is $f_k(p^a)$, which follows from (1.3) and (1.19). This establishes (2.29).

In summary, we arrive at the following conjecture.

Conjecture 2.2. *(Additive divisor conjecture) Let $k > 1$ be a natural number and $x > 0$ is large. Then there exists a positive constant $\theta_k \in [\frac{1}{2}, 1)$ such that*

$$D_k(x, h) = \int_0^x q_k(t, h)dt + E_k(x, h), \tag{2.35}$$

where $q_k(t, h)$ is given by (2.22) and for every $\varepsilon > 0$

$$E_k(x, h) \ll x^{\vartheta_k + \varepsilon} \text{ uniformly for } 1 \le h \le x^{1-\varepsilon}. \tag{2.36}$$

Moreover, in (2.22), the coefficient of $\log(t) \log(t + h)$ is $\mathcal{D}_k(0, 0) = C_k f_k(h)$ where C_k is given by (1.3) and $f_k(h)$ is the multiplicative function defined by (1.4).

Remarks.

(1) It appears that Titchmarsh [45] was the first to conjecture the leading term in the asymptotic formula for a weighted version of $D_3(x, 1)$, based on the circle method. Vinogradov [49] proposed the general form of a conjectural formula for $D_3(x, h)$ (see equation (2) of [49]). However, few details were given and he did not provide any formulae for the coefficients of $q_k(t, h)$. Then in the nineties Ivic [28], [29] and Conrey-Gonek [9] provided more precise formulae following Duke, Friedlander, and Iwaniec's δ-method.

(2) It is not clear what is the true size of the error term $E_k(x, h)$ and various opinions have been expressed. Conrey and Gonek [9] conjectured that $\vartheta_k = \frac{1}{2}$ in the case that $h \le \sqrt{x}$. However, Conrey and Keating [10] revised this to $\vartheta_k = \frac{1}{2}$ is valid for all $h \le x^{1-\varepsilon}$. Recent work of Farzad Aryan suggests that $\vartheta_2 = \frac{1}{2}$ is the correct value. In fact, Aryan [2] shows that a smoothed variant of $D_2(x, h)$ has error term $O(x^{\frac{1}{2}+\varepsilon} h^{\alpha})$ where α is given by (2.8). On the other hand, Vinogradov [49] conjectured that $E_k(x, h) \ll x^{1-\frac{1}{k}}$ in the case of h *fixed*. Ivic [28] suggested that Vinogradov's bound was slightly too strong and that perhaps $E_k(x, h) \ll x^{1-\frac{1}{k}} (\log x)^{D_k}$ for a positive constant D_k. In light of these diverging opinions, it would be beneficial to have numerical data checking this conjecture.

(3) Note that the conjecture is sometimes written as

$$D_k(x,h) = \int_0^x r_k(t,h)dt + \tilde{E}_k(x,h),$$ (2.37)

where $\tilde{E}_k(x,h) \ll x^{\vartheta_k + \varepsilon}$ uniformly for $1 \le h \le x^{1-\varepsilon}$. By (2.16) we may replace $q_k(t,h)$ by $r_k(t,h)$ with an error $O(x^{\frac{1}{2}+\varepsilon})$ for $h \le \sqrt{x}$. Since we expect that $\theta_k \ge \frac{1}{2}$, it should not matter whether the main term in (2.35) or (2.37) is used for $h \le \sqrt{x}$. However, as we expect to have an asymptotic formula for $h \le x^{1-\varepsilon}$, it is preferable to use the form (2.35).

(4) In the case $k=2$, this conjecture agrees with Ingham's result (2.1). Note that $2(1-\frac{1}{p}) - (1-\frac{1}{p})^2 = 1 - p^{-2}$ and thus $C_2 = \prod_p(1-p^{-2}) = \frac{6}{\pi^2}$. Also, $\tau_1(p^j) = \tau_1(p^{j+1}) = 1$, $H_{1,j}(u) = H_{1,j+1}(u) = 1$, and by (1.4) $f_2(p^\alpha) = \frac{\sum_{j=0}^\alpha (\frac{1}{p^j} - \frac{1}{p^{j+2}})}{1-p^{-2}} = \sum_{j=0}^\alpha \frac{1}{p^j} = \sigma_{-1}(p^\alpha)$.

(5) Recently, Andrade, Bary-Soker, and Rudnick [1] proved a function field version of the above conjecture.

(6) Although Conjecture 2.2 remains open for $k \ge 3$, averaged versions have been established. For instance, see [3] and [30].

3. A LOWER BOUND FOR $D_k(x,h)$

In this section, we establish Theorem 1.1, which provides a lower bound for $D_k(x,h)$. Before proving this result, we require a proposition which gives an asymptotic estimate for a certain divisor sum.

Proposition 3.1. *Let $\ell \in \mathbb{N}$ and $\ell \ge 2$.*
(i) Then there exists $h_0 = h_0(\ell) > 0$ such that for $h \ge h_0$,

$$\sum_{\substack{a,b \le X \\ (a,b)|h}} \frac{\tau_\ell(a)\tau_\ell(b)}{[a,b]} = \frac{\tilde{C}_\ell g_\ell(h)}{(\ell!)^2}(\log X)^{2\ell} + O_\ell\left(\prod_{p|h}(1+p^{-1})^{\ell^2}(\log X)^{2\ell-1}\log\log h\right)$$ (3.1)

$$+ O_\ell\left(\exp\left((12.94\ell^4 - 12.81\ell^3 + 4.52\ell^2)\frac{(\log h)^{1-0.99/\ell}}{\log\log h}\right)(\log X)^{2\ell}X^{-0.99/\ell}\right)$$

where

$$\tilde{C}_\ell = \prod_p\left(2\left(1-\frac{1}{p}\right)^\ell - \left(1-\frac{1}{p}\right)^{2l}\right),$$ (3.2)

and g_ℓ is the multiplicative function defined on prime powers by

$$g_\ell(p^\alpha) := \sum_{j=0}^\alpha \left(\frac{\sigma_\ell(p^j,1)^2}{p^j} - \frac{\sigma_\ell(p^{j+1},1)^2}{p^{j+2}}\right)\left(2\left(1-\frac{1}{p}\right)^\ell - \left(1-\frac{1}{p}\right)^{2l}\right)^{-1}.$$ (3.3)

(ii) If $1 \le h < h_0$, then the same result holds as in equation (3.1) except the second O_ℓ term in this equation is replaced by $O_\ell((\log X)^{2\ell}X^{-0.99/\ell})$ where O_ℓ constant is polynomial in ℓ.

We have not tried to obtain the best possible error term here. Note that the sum in this proposition bears some resemblance to the quadratic forms that occur in the standard Selberg sieve [41]. A similar sum is studied in [15].

With these two results in hand, we prove our main result.

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Proof of Theorem 1.1. Let $x > 1$. For the lower bound, we make use of the identity

(3.4)
$$\tau_k(m) \geq \sum_{\substack{d|m \\ d \leq \sqrt{x}}} \tau_{k-1}(d) \text{ for } m \geq x.$$

It follows that

$$\sum_{x \leq n \leq 2x} \tau_k(n)\tau_k(n+h) \geq \sum_{x \leq n \leq 2x} \sum_{\substack{a|n \\ a \leq \sqrt{x}}} \tau_{k-1}(a) \sum_{\substack{b|n+h \\ b \leq \sqrt{x+h}}} \tau_{k-1}(b)$$

$$= \sum_{a \leq \sqrt{x}} \sum_{b \leq \sqrt{x+h}} \tau_{k-1}(a)\tau_{k-1}(b) \sum_{\substack{x \leq n \leq 2x \\ a|n \\ b|n+h}} 1.$$

If $(a, b) \mid h$, the inner sum is $\frac{x}{[a,b]} + O(1)$ and otherwise it is 0. Thus

$$\sum_{x \leq n \leq 2x} \tau_k(n)\tau_k(n+h) \geq \sum_{a \leq \sqrt{x}} \sum_{\substack{b \leq \sqrt{x} \\ (a,b)|h}} \tau_{k-1}(a)\tau_{k-1}(b)\left(\frac{x}{[a,b]} + O(1)\right).$$

The $O(1)$ term contributes

$$\left(\sum_{a \leq \sqrt{x}} \tau_{k-1}(a)\right)^2 \ll (\sqrt{x}(\log x)^{k-2})^2 \ll x(\log x)^{2k-4}$$

and by Proposition 3.1 with $X = \sqrt{x}$

(3.5)
$$\sum_{\substack{a,b \leq \sqrt{x} \\ (a,b)|h}} \frac{\tau_{k-1}(a)\tau_{k-1}(b)}{[a,b]} = \frac{\tilde{C}_{k-1}g_{k-1}(h)}{(k-1)!^2 2^{2k-2}}(\log x)^{2k-2}$$

$$+ O_k\left(g_{k-1}(h)(\log x)^{2k-3}\log\log h + \exp\left(\frac{\mathcal{C}_k(\log h)^{\vartheta_k}}{\log\log h}\right)(\log x)^{2k-2}x^{-\beta_k}\right)$$

where $\tilde{C}_{k-1}$ is defined by (3.2), $\mathcal{C}_k = \frac{6.41}{C}(k-1)^3 + 4.56(k-1)^2$, $\vartheta_k = 1 - \frac{0.99}{k-1}$, and $\beta_k = \frac{0.495}{k-1}$. It may be checked that $g_{k-1}(h) \gg_k 1$ for all $h \in \mathbb{N}$. The second error term in (3.5) is dominated by the first if and only if $\frac{x^{\beta_k}}{\log x} \gg \frac{\exp(\mathcal{C}_k \frac{(\log h)^{\vartheta_k}}{\log\log h})}{\log\log h}$. In other words,

(3.6)
$$\exp(\beta_k \log x - \log_2 x) \gg \exp\left(\frac{\mathcal{C}_k(\log h)^{\vartheta_k}}{\log\log h} - \log_3 h\right).$$

This inequality will hold if we impose the condition $\frac{\mathcal{C}_k(\log h)^{\vartheta_k}}{\log\log h} \leq \frac{\beta_k}{2}\log x$. This implies that $\log_2 h \ll \log_2 x$. Therefore $\mathcal{C}_k(\log h)^{\vartheta_k} \leq \frac{\beta_k}{2}\log x \log\log x$. Solving for h we find that $h \leq \exp(\tilde{B}_k(\log x \log\log x)^{\frac{k-1}{k-1.99}})$ for some positive $\tilde{B}_k$. Combining the above,

(3.7)
$$\sum_{x \leq n \leq 2x} \tau_k(n)\tau_k(n+h) \geq \frac{\tilde{C}_{k-1}g_{k-1}(h)}{(k-1)!^2 2^{2k-2}}x(\log x)^{2k-2} + O_k(g_{k-1}(h)x(\log x)^{2k-3}\log\log h),$$

as long as $h \leq \exp(\tilde{B}_k(\log x \log\log x)^{1/\vartheta_k})$. Now split the interval $[\sqrt{x}, x]$ into $O(\log x)$ dyadic intervals and apply (3.7) to obtain

(3.8)
$$\sum_{\sqrt{x} \leq n \leq x} \tau_k(n)\tau_k(n+h) \geq \frac{C_k f_k(h)}{(k-1)!^2 2^{2k-2}}x(\log x)^{2k-2} + O_k(f_k(h)x(\log x)^{2k-3}\log\log h)$$

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valid for $h \le \exp(B_k(\log x \log \log x)^{\frac{k-1}{k-1.99}})$ for another positive constant B_k, where $C_k := \tilde{C}_{k-1}$ and $f_k(h) := g_{k-1}(h)$. Since τ_k is a positive function, we establish the theorem. $\qquad\square$

Remark. The above argument in the case $k = 2$ yields an asymptotic formula for $D_2(x, h)$. This is essentially the argument Ingham used in [26] and [27] to obtain first an upper bound and then an asymptotic for $D_2(x, h)$.

We have reduced the proof of Theorem 1.1 to a verification of Proposition 3.1. Not surprisingly, we must understand the double Dirichlet series

$$\mathcal{A}(s_1, s_2) = \sum_{\substack{a,b \ge 1 \\ (a,b)|h}} \frac{\tau_\ell(a)\tau_\ell(b)}{[a,b]a^{s_1}b^{s_2}}.$$

We shall show that $\mathcal{A}(s_1, s_2) = \zeta(s_1 + 1)^\ell \zeta(s_2 + 1)^\ell \mathcal{B}(s_1, s_2)$ where

$$(3.9) \qquad \mathcal{B}(s_1, s_2) = \sum_{g|h} \frac{1}{g^{s_1+s_2+1}} \sum_{d=1}^{\infty} \frac{\mu(d)\sigma_\ell(gd, s_1 + 1)\sigma_\ell(gd, s_2 + 1)}{d^{s_1+s_2+2}}$$

and we recall that σ_ℓ is the multiplicative function defined by $\sigma_\ell(n, s) = \left(\sum_{a=1}^{\infty} \frac{\tau_\ell(na)}{a^s} \right) \zeta(s)^{-\ell}$. (Some properties of σ_ℓ are listed in subsection 1.1.) We require the following bounds on $\mathcal{B}(s_1, s_2)$.

Lemma 3.2. *For $z \in \mathbb{C}$ and $h \in \mathbb{N}$, set*

$$(3.10) \qquad \Theta(z, h) = \prod_{p|h}(1 + p^{-z})^{\ell^2}.$$

(i) Let $\sigma_1 = \Re(s_1)$ and $\sigma_2 = \Re(s_2)$. Then

$$(3.11) \qquad |\mathcal{B}(s_1, s_2)| \ll \Theta(\sigma_1 + \sigma_2 + 1, h) \text{ for } \sigma_1, \sigma_2 \ge -0.99, \sigma_1 + \sigma_2 \ge -0.99.$$

(ii) We have

$$(3.12) \qquad \mathcal{B}(0, 0) = \tilde{C}_\ell g_\ell(h).$$

(iii)

$$(3.13) \qquad \mathcal{B}^{(i_1, i_2)}(0, 0) \ll_\ell \Theta(1, h)(\log \log h)^{i_1 + i_2}.$$

We also require a bound for a certain zeta integral.

Lemma 3.3. *Let $0 < \varepsilon < 1$, $\ell \in \mathbb{N}$, $s \in \mathbb{C}$ with $\Re(s) \ge -\frac{1}{\ell}$, then*

$$(3.14) \qquad \int_{-\infty}^{\infty} \min\left(\frac{1}{|s|}, \frac{\varepsilon^{-1}}{|s(s+1)|} \right) |\zeta(s+1)|^\ell dt \ll_\ell \varepsilon^{-1} \text{ where } s = \sigma + it.$$

The next lemma is used to bound $\Theta(z, h)$ when $\Re(z) < 1$.

Lemma 3.4. *Let $\kappa \in [0.5, 1)$. There exists $x_\kappa > 0$ such that if $x \ge x_\kappa$, then*

$$(3.15) \qquad \sum_{p \le x} p^{-\kappa} \le \left(\frac{12.68\kappa}{(1-\kappa)^2} + 3.17 \right) \frac{x^{1-\kappa}}{\log x}.$$

Proof. By Theorem 1 of [40] it follows that

$$(3.16) \qquad \pi(x) \le \frac{3.17x}{\log x} \text{ for } x \ge 2.$$

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By partial summation

$$\sum_{2 \leq p \leq x} p^{-\kappa} = \frac{\pi(t)}{t^\kappa}\Big|_2^x + \kappa \int_2^x \frac{\pi(t)}{t^{\kappa+1}} dt \leq \frac{\pi(x)}{x^\kappa} + \kappa \int_2^x \frac{\pi(t)}{t^{\kappa+1}} dt$$

(3.17)

$$\leq 3.17 \Big(\frac{x^{1-\kappa}}{\log x} + \kappa \int_2^x \frac{1}{t^\kappa \log t} dt \Big),$$

by (3.16). We now bound the integral. Let $y \in (2, x)$ and thus

$$\int_2^x \frac{1}{t^\kappa \log t} dt = \int_2^y \frac{1}{t^\kappa \log t} dt + \int_y^x \frac{1}{t^\kappa \log t} dt$$

$$\leq \frac{y-2}{2^\kappa \log 2} + \frac{1}{\log y} \int_y^x t^{-\kappa} dt$$

$$\leq \frac{y}{\sqrt{2} \log 2} + \frac{x^{1-\kappa}}{(1-\kappa) \log y}.$$

For x sufficiently large, there exists $y \in (2, x)$ such that

(3.18)
$$\frac{y}{\sqrt{2} \log 2} = \frac{x^{1-\kappa}}{(1-\kappa) \log y}.$$

Moreover, (3.18) implies that $\log y > \frac{1-\kappa}{2} \log x$. Thus for $x \gg_\kappa 1$, we have

$$\sum_{2 \leq p \leq x} p^{-\kappa} \leq 3.17 \Big(\frac{4\kappa}{(1-\kappa)^2} + 1 \Big) \frac{x^{1-\kappa}}{\log x}$$

and we obtain (3.15). $\qquad \square$

With these lemmas in hand, we now establish Proposition 3.1.

Proof of Proposition 3.1. We shall give the proof in the case $h \geq 2$. At the end of the proof we will discuss the modifications required in the simpler case $h = 1$. A standard approach would be to apply Perron's formula twice. Instead, we find it simpler to smooth the truncated sum. To simplify the evaluation of the previous sum, we insert smoothing factors. Let η be positive and let $\epsilon \in (0, 1)$ be a small positive number. Let $\phi = \phi_{\eta,\epsilon}(t)$ denote a smooth, non-negative function such that

(3.19)
$$\phi_{\eta,\epsilon}(t) = \begin{cases} 1 & \text{if } t \in [0, \eta], \\ 0 & \text{if } t \in [\eta + \epsilon, \infty). \end{cases}$$

Observe that the support of ϕ is contained in $[0, \eta + \epsilon]$. We also require the derivatives to satisfy

(3.20)
$$\phi_{\eta,\epsilon}^{(j)}(t) \ll \epsilon^{-j}.$$

Later, we shall choose the parameter η to be either $1 - \epsilon$ or 1.

We shall evaluate sums of the form

$$\mathscr{I}(\phi) = \sum_{\substack{a,b \in \mathbb{N} \\ (a,b)|h}} \frac{\tau_\ell(a)\tau_\ell(b)}{[a,b]} \phi\Big(\frac{a}{X}\Big) \phi\Big(\frac{b}{X}\Big).$$

where $\phi(t) = \phi_{\eta,\epsilon}(t)$. We define the Mellin transform

(3.21)
$$\Phi(s) = \int_0^\infty \phi(t) t^{s-1} dt.$$

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This is absolutely convergent for $\Re(s) > 0$. By Mellin inversion, we have

(3.22)
$$\phi(t) = \frac{1}{2\pi i} \int_{(c)} \Phi(s) t^{-s} ds$$

where $c > 0$. By two applications of (3.22)

(3.23)
$$\mathscr{I}(\phi) = \frac{1}{(2\pi i)^2} \int_{(c_1)} \int_{(c_2)} \mathcal{A}(s_1, s_2) X^{s_1+s_2} \Phi(s_1) \Phi(s_2) ds_1 ds_2$$

where $c_1, c_2 > 0$, and

$$\mathcal{A}(s_1, s_2) = \sum_{\substack{a,b \geq 1 \\ (a,b)|h}} \frac{\tau_\ell(a) \tau_\ell(b)}{[a,b] a^{s_1} b^{s_2}}.$$

The general approach to evaluate (3.23) is to move each of the contours to the left of $\Re(s_1) = 0$ and $\Re(s_2) = 0$ and apply the residue theorem. The integrand in (3.23) has poles at $s_1 = 0$ and $s_2 = 0$ arising from $\mathcal{A}(s_1, s_2)$ and from $\Phi(s_1)$ and $\Phi(s_2)$. A main term will arise from these poles. The new contours will contribute an error term. In order to evaluate the residue and the error terms we need to understand the behaviour of $\mathcal{A}(s_1, s_2)$, $\Phi(s_1)$, and $\Phi(s_2)$ near the poles at $s_1 = 0$ and $s_2 = 0$ and we need to provide bounds for these functions when $\Im(s_1)$ and $\Im(s_2)$ are large. First, we consider the behaviour of $\Phi(s)$. By an integration by parts, it follows that

(3.24)
$$\Phi(s) = \frac{1}{s} \Psi(s)$$

where

(3.25)
$$\Psi(s) = -\int_0^\infty \phi'(t) t^s dt.$$

This is originally valid for $\Re(s) > 0$. However, it is clear that $\Psi(s)$ is entire function. Thus $\Phi(s)$ is holomorphic everywhere on $\mathbb{C}$ with the exception of a simple pole at $s = 0$. Note that we have the Laurent expansion

(3.26)
$$\Phi(s) = \frac{\Psi(0)}{s} + \Psi'(0) + \frac{\Psi''(0)}{2} s + \cdots.$$

We shall require some bounds for the expressions $\Psi^{(j)}(0)$. Observe that

(3.27)
$$\Psi^{(j)}(0) = -\int_\eta^{\eta+\epsilon} \phi'(t) (\log t)^j dt.$$

Therefore

(3.28)
$$\Psi(0) = \int_\eta^{\eta+\epsilon} \phi'(t) dt = \phi(\eta) = 1$$

and

(3.29)
$$|\Psi^{(j)}(0)| \leq \int_\eta^{\eta+\epsilon} |\phi'(t)| \max_{\eta \leq t \leq \eta+\epsilon} |\log t|^j dt \ll \epsilon^j.$$

Integrating (3.21) by parts m times, we find that

$$\Phi(s) = \frac{(-1)^m}{s(s+1)\cdots(s+m-1)} \int_0^\infty \phi^{(m)}(t) t^{s+m-1} dt,$$

which is valid for all $s \in \mathbb{C} \setminus \{0\}$. Note that for $m \geq 2$ the integrand has simple zeros at $s = -1, \ldots, -(m-1)$. Thus for $m \geq 1$ and $s \in \mathbb{C} \setminus \{0, -1, \ldots, -(m-1)\}$,

$$(3.30) \qquad |\Phi(s)| \leq \frac{1}{|s(s+1)\cdots(s+m-1)|} \int_{\eta}^{\eta+\epsilon} |\phi^{(m)}(t)| t^{\sigma+m-1} dt \ll_m \frac{\epsilon^{1-m}(\eta+\epsilon)^{\sigma+m-1}}{|s(s+1)\cdots(s+m-1)|}.$$

Next, we simplify the Dirichlet series $\mathcal{A}(s_1, s_2)$. We let $g = (a, b)$ and make the variable change $a = gc$, $b = gd$ with $(c, d) = 1$, and group terms according to $g \mid h$

$$\mathcal{A}(s_1, s_2) = \sum_{g|h} \sum_{\substack{a,b \geq 1 \\ (a,b)=g}} \frac{\tau_\ell(a)\tau_\ell(b)}{[a,b] a^{s_1} b^{s_2}} = \sum_{g|h} g \sum_{\substack{a,b \geq 1 \\ (a,b)=g}} \frac{\tau_\ell(a)\tau_\ell(b)}{a^{s_1+1} b^{s_2+1}}$$

$$= \sum_{g|h} \frac{1}{g^{s_1+s_2+1}} \sum_{\substack{c,d \geq 1 \\ (c,d)=1}} \frac{\tau_\ell(gc)\tau_\ell(gd)}{c^{s_1+1} d^{s_2+1}}.$$

The condition $(c, d) = 1$ is detected by $\sum_{e|c, e|d} \mu(e)$ and thus

$$(3.31) \qquad \mathcal{A}(s_1, s_2) = \sum_{g|h} \frac{1}{g^{s_1+s_2+1}} \sum_{e=1}^{\infty} \frac{\mu(e)}{e^{s_1+s_2+2}} \sum_{c,d \geq 1} \frac{\tau_\ell(gec)\tau_\ell(ged)}{c^{s_1+1} d^{s_2+1}}.$$

Inserting (1.14) in (3.31), it follows that

$$(3.32) \qquad \mathcal{A}(s_1, s_2) = \mathcal{B}(s_1, s_2)\zeta(s_1+1)^\ell \zeta(s_2+1)^\ell$$

where

$$(3.33) \qquad \mathcal{B}(s_1, s_2) = \sum_{g|h} \frac{1}{g^{s_1+s_2+1}} \sum_{e=1}^{\infty} \frac{\mu(e)\sigma_\ell(ge, s_1+1)\sigma_\ell(ge, s_2+1)}{e^{s_1+s_2+2}}.$$

By Fubini's theorem, we have

$$\mathscr{I}(\phi) = \frac{1}{(2\pi i)^2} \int_{(c_2)} \int_{(c_1)} \mathcal{B}(s_1, s_2)\zeta(s_1+1)^\ell \zeta(s_2+1)^\ell X^{s_1+s_2} \Phi(s_1)\Phi(s_2) ds_1 ds_2.$$

The evaluation of multiple integrals of this type is now standard. For instance, in [21] and [7] more complicated integrals are treated. Note that the main term shall arise from the pole of order ℓ at $s_1 = 0$ and the pole of order ℓ at $s_2 = 0$ of the integrand. For each fixed s_2 with $\Re(s_2) = c_2$, the residue theorem implies that

$$(3.34) \qquad \frac{1}{2\pi i} \int_{(c_1)} \mathcal{B}(s_1, s_2)\Phi(s_1)\zeta(s_1+1)^\ell X^{s_1} ds_1 = \mathrm{Res}_{s_1=0}\Big(\mathcal{B}(s_1, s_2)\Phi(s_1)\zeta(s_1+1)^\ell X^{s_1}\Big) + g(s_2)$$

where

$$(3.35) \qquad g(s_2) = \frac{1}{2\pi i} \int_{(c_1')} \mathcal{B}(s_1, s_2)\Phi(s_1)\zeta(s_1+1)^\ell X^{s_1} ds_1$$

and $-1 < c_1' < 0$. By the Laurent expansions (3.26),

$$(3.36) \qquad \mathcal{B}(s_1, s_2) = \mathcal{B}(0, s_2) + \mathcal{B}^{(1,0)}(0, s_2)s_1 + \frac{1}{2!}\mathcal{B}^{(2,0)}(0, s_2)s_1^2 + \cdots$$

$$(3.37) \qquad \zeta(s_1+1)^\ell = s_1^{-\ell}(\alpha_{0,\ell} + \alpha_{1,\ell}s_1 + \alpha_{2,\ell}s_1^2 + \cdots), \text{ where } \alpha_{0,\ell} = 1,$$

$$(3.38) \qquad X^{s_1} = 1 + (\log X)s_1 + \tfrac{1}{2}(\log X)^2 s_1^2 + \cdots$$

it follows that

$$(3.39) \qquad \mathrm{Res}_{s_1=0}\Big(\mathcal{B}(s_1,s_2)\Phi(s_1)\zeta(s_1+1)^\ell X^{s_1}\Big) = \sum_{\substack{i_1+i_2+i_3+i_4=\ell \\ i_1,i_2,i_3,i_4\geq 0}} \frac{\mathcal{B}^{(i_1,0)}(0,s_2)\Psi^{(i_2)}(0)\alpha_{i_3,\ell}(\log X)^{i_4}}{i_1!i_2!i_4!}.$$

We now bound $g(s_2)$:

$$(3.40) \qquad |g(s_2)| \leq \Theta(c_1'+\sigma_2+1,h)X^{c_1'}\int_{-\infty}^{\infty}\min\Big(\frac{1}{|s_1|},\frac{\epsilon^{-1}}{|s_1(s_1+1)|}\Big)|\zeta(s_1+1)|^\ell dt_1 \text{ where } s_1 = c_1' + it_1.$$

It follows from Lemma 3.3 with $c_1' \geq -1/\ell$

$$(3.41) \qquad g(s_1) \ll \epsilon^{-1}\Theta(c_1'+\sigma_2+1,h)X^{c_1'}.$$

Thus we have

$$(3.42) \qquad \begin{aligned} \mathscr{I}(\phi) &= \frac{1}{2\pi i}\int_{(c_2)}\zeta(s_2+1)^\ell X^{s_2}\Phi(s_2)\mathrm{Res}_{s_1=0}\Big(\mathcal{B}(s_1,s_2)\Phi(s_1)\zeta(s_1+1)^\ell X^{s_1}\Big)ds_2 \\ &+ \frac{1}{2\pi i}\int_{(c_2)}\zeta(s_2+1)^\ell X^{s_2}\Phi(s_2)g(s_2)ds_2. \end{aligned}$$

By (3.30) and (3.41) the second integral is bounded by

$$(3.43) \qquad \begin{aligned} &\epsilon^{-1}\Theta(c_1'+c_2+1,h)X^{c_1'}X^{c_2}\int_{-\infty}^{\infty}\min\Big(\frac{1}{|s_2|},\frac{\epsilon^{-1}}{|s_2(s_2+1)|}\Big)dt_2 \text{ where } s_2 = c_2 + it_2 \\ &\ll \epsilon^{-2}\Theta(c_1'+c_2+1,h)X^{c_1'+c_2}, \end{aligned}$$

by another application of Lemma 3.3. Choosing $c_1' = -1/\ell$ and $c_2 = 0.01/\ell$, it follows that

$$(3.44)$$
$$\mathscr{I}(\phi) = \frac{1}{2\pi i}\int_{(c_2)}\zeta(s_2+1)^\ell X^{s_2}\Phi(s_2)\mathrm{Res}_{s_1=0}\Big(\mathcal{B}(s_1,s_2)\Phi(s_1)\zeta(s_1+1)^\ell X^{s_1}\Big)ds_2 + O_\ell(\Theta(1-\tfrac{0.99}{\ell},h)\epsilon^{-2}X^{-0.99/\ell}).$$

By (3.39) we see that

$$(3.45) \qquad \mathscr{I}(\phi) = \sum_{\substack{i_1+i_2+i_3+i_4=\ell \\ i_1,i_2,i_3,i_4\geq 0}} \frac{\Psi^{(i_2)}(0)\alpha_{i_3,\ell}(\log X)^{i_4}}{i_1!i_2!i_4!}\mathscr{I}_{i_1}(\phi) + O_\ell(\Theta(1-\tfrac{0.99}{\ell},h)\epsilon^{-2}X^{-0.99/\ell}).$$

where

$$(3.46) \qquad \mathscr{I}_{i_1}(\phi) = \frac{1}{2\pi i}\int_{(c_2)}\mathcal{B}^{(i_1,0)}(0,s_2)\zeta(s_2+1)^\ell X^{s_2}\Phi(s_2)ds_2 \text{ for } i_1 \geq 0.$$

By an application of the residue theorem,

$$(3.47) \quad \mathscr{I}_{i_1}(\phi) = \mathrm{Res}_{s_2=0}\Big(\mathcal{B}^{(i_1,0)}(0,s_2)\Phi(s_2)\zeta(s_2+1)^\ell X^{s_2}\Big) + \frac{1}{2\pi i}\int_{(c_2')}\mathcal{B}^{(i_1,0)}(0,s_2)\Phi(s_2)\zeta(s_2+1)^\ell X^{s_2}ds_2.$$

The second integral can be evaluated very similarly to $g(s_2)$. However, we require a bound for $\mathcal{B}^{(i_1,0)}(0,s_2)$ with $\sigma_2 = c_2'$. By Cauchy's integral formula

$$(3.48) \qquad \mathcal{B}^{(i_1,0)}(0,s_2) = \frac{i_1!}{2\pi i}\int_{|z-s_2|=\delta}\frac{\mathcal{B}(0,z)}{(z-s_2)^{i_1+1}}dz$$

where $\delta > 0$. By an application of Lemma 3.2, (3.11) it follows that

$$(3.49) \qquad \mathcal{B}^{(i_1,0)}(0,s_2) \ll \Theta(c_2'-\delta+1,h)\delta^{-i_1},$$

as long as $c_2' - \delta \geq -0.99$. Therefore, by the above bound and Lemma 3.3

$$\frac{1}{2\pi i} \int_{(c_2')} \mathcal{B}^{(i_1,0)}(0, s_2)\Phi(s_2)\zeta(s_2+1)^\ell X^{s_2} ds_2$$

(3.50)

$$\ll \Theta(c_2' - \delta + 1, h)\delta^{-i_1} X^{c_2'} \int_{-\infty}^\infty \min\left(\frac{1}{|s_2|}, \frac{\epsilon^{-1}}{|s_2(s_2+1)|}\right)|\zeta(s_2+1)|^\ell ds_2$$

$$\ll \epsilon^{-1}\Theta(c_2' - \delta + 1, h)\delta^{-i_1} X^{c_2'}$$

$$\ll_\ell \epsilon^{-1}\Theta(1 - \tfrac{0.99}{\ell}, h)X^{-1/\ell},$$

by the choices $c_2' = -1/\ell$ and $\delta = 0.01/\ell$. Thus

(3.51) $$\mathscr{I}_{i_1}(\phi) = \mathrm{Res}_{s_2=0}\left(\mathcal{B}^{(i_1,0)}(0, s_2)\Phi(s_2)\zeta(s_2+1)^\ell X^{s_2}\right) + O_\ell(\epsilon^{-1}\Theta(1 - \tfrac{0.99}{\ell}, h)X^{-1/\ell}).$$

Computing the residue in (3.51) gives

$$\mathscr{I}_{i_1}(\phi) = \sum_{\substack{j_1+j_2+j_3+j_4=\ell \\ j_1,j_2,j_3,j_4 \geq 0}} \frac{\mathcal{B}^{(i_1,j_1)}(0,0)\Psi^{(j_2)}(0)\alpha_{j_3,\ell}(\log X)^{j_4}}{j_1!j_2!j_4!} + O_\ell(\epsilon^{-1}\Theta(1 - \tfrac{0.99}{\ell}, h)X^{-1/\ell}).$$

Inserting this last expression in (3.45) yields

$$\mathscr{I}(\phi) = \sum_{\substack{i_1+i_2+i_3+i_4=\ell \\ i_1,i_2,i_3,i_4 \geq 0}} \frac{\Psi^{(i_2)}(0)\alpha_{i_3,\ell}(\log X)^{i_4}}{i_1!i_2!i_4!} \sum_{\substack{j_1+j_2+j_3+j_4=\ell \\ j_1,j_2,j_3,j_4 \geq 0}} \frac{\mathcal{B}^{(i_1,j_1)}(0,0)\Psi^{(j_2)}(0)\alpha_{j_3,\ell}(\log X)^{j_4}}{j_1!j_2!j_4!}$$

$$+ O_\ell\left(\left(\sum_{i_2+i_4=\ell} \epsilon^{i_2}(\log X)^{i_4}\right)\epsilon^{-1}\Theta(1 - \tfrac{0.99}{\ell}, h)X^{-1/\ell} + \epsilon^{-2}\Theta(1 - \tfrac{0.99}{\ell}, h)X^{-0.99/\ell}\right),$$

where we have used $\Psi^{(i_2)}(0) \ll \epsilon^{i_2}$ and $\alpha_{i_3,\ell} = O_\ell(1)$. The sum in the big O term is bounded by $(\log X)^\ell$ as $\epsilon < 1$. The main contribution to $\mathscr{I}(\phi)$ is $\mathcal{B}(0,0)(\log X)^{2\ell}/(\ell!)^2$ which arises from $(i_1, i_2, i_3, i_4) = (j_1, j_2, j_3, j_4) = (0, 0, 0, \ell)$. By (3.85) and (3.29) the remaining terms are bounded by

$$\ll_\ell \sum_{\substack{i_1+i_2+i_3+i_4=\ell \\ j_1+j_2+j_3+j_4=\ell}}{}' \frac{\Theta(1, h)(\log\log h)^{i_1+j_1}\epsilon^{i_2+j_2}(\log X)^{i_4+j_4}}{i_1!i_2!i_4!j_1!j_2!j_4!}$$

where $'$ in the summation indicates that the terms $(i_1, i_2, i_3, i_4) = (0, 0, 0, \ell)$ and $(j_1, j_2, j_3, j_4) = (0, 0, 0, \ell)$ have been excluded. Since $\epsilon < 1$ and either i_4 or $j_4 \leq \ell - 1$, it follows that the remaining terms are bounded by

$$\ll_\ell \Theta(1, h) \sum_{\alpha+\beta \leq 2\ell-1} (\log\log h)^\alpha (\log X)^\beta \ll_\ell \Theta(1, h)(\log X)^{2\ell-1} \log\log h.$$

Combining the above facts, we find

(3.52)

$$\mathscr{I}(\phi) = \frac{\tilde{C}_\ell g_\ell(h)}{(\ell!)^2}(\log X)^{2\ell} + O_\ell(\Theta(1, h)(\log X)^{2\ell-1} \log\log h + ((\log X)^\ell \epsilon^{-1} + \epsilon^{-2})\Theta(1 - \tfrac{0.99}{\ell}, h)X^{-0.99/\ell}).$$

We now remove the smooth weight to obtain an asymptotic formula for the truncated sum. Let

(3.53) $$\phi_-(t) = \phi_{1-\epsilon,\epsilon}(t) \text{ and } \phi_+(t) = \phi_{1,\epsilon}(t)$$

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be the functions corresponding to the choices $\eta = 1 - \epsilon$ and $\eta = 1$. Note that $\phi^-(t)$ and $\phi^+(t)$ are a smooth minorant and majorant of $\mathbb{1}_{[0,1]}(t)$, the indicator function of $[0, 1]$. It follows that

$$(3.54) \qquad \mathscr{I}(\phi_-) \leq \sum_{\substack{a,b \leq X \\ (a,b)|h}} \frac{\tau_\ell(a)\tau_\ell(b)}{[a,b]} \leq \mathscr{I}(\phi_+).$$

From (3.52) and (3.54) we have

$$(3.55)$$

$$\sum_{\substack{a,b \leq X \\ (a,b)|h}} \frac{\tau_\ell(a)\tau_\ell(b)}{[a,b]} = \frac{\tilde{C}_\ell g_\ell(h)}{(\ell!)^2}(\log X)^{2\ell} + O_\ell(\Theta(1,h)(\log X)^{2\ell-1}\log\log h + ((\log X)^\ell \epsilon^{-1} + \epsilon^{-2})\Theta(1 - \tfrac{0.99}{\ell}, h)X^{-0.99/\ell})$$

$$= \frac{\tilde{C}_\ell g_\ell(h)}{(\ell!)^2}(\log X)^{2\ell} + O_\ell(\Theta(1,h)(\log X)^{2\ell-1}\log\log h + \Theta(1 - \tfrac{0.99}{\ell}, h)(\log X)^{2\ell}X^{-0.99/\ell}),$$

by the choice $\epsilon = (\log X)^{-\ell}$. Finally, we bound $\Theta(\kappa, h)$ where $\kappa = 1 - \frac{0.99}{\ell}$. We have

$$\log\Theta(\kappa, h) = \ell^2 \sum_{p|h}\log(1 + p^{-\kappa}) \leq \ell^2 \sum_{p|h} p^{-\kappa}$$

since $\log(1 + x) \leq x$ for $x \geq 0$. Let $\omega(h)$ denote the number of prime divisors of h. If $h \geq h_0(\ell)$, then

$$(3.56)$$

$$\log\Theta(\kappa, h) \leq \ell^2\left(\sum_{p \leq \log h} p^{-\kappa} + \frac{1}{(\log h)^\kappa}\omega(h)\right)$$

$$\leq \ell^2\left(\left(\frac{12.68\kappa}{(1-\kappa)^2} + 3.17\right)\frac{(\log h)^{1-\kappa}}{\log\log h} + \frac{1.3841(\log h)^{1-\kappa}}{\log\log h}\right)$$

by (3.15) and Théorème 11 of [39, Robin]. It follows that

$$(3.57) \qquad \Theta(1 - \tfrac{0.99}{\ell}, h) \leq \exp\left((12.94\ell^4 - 12.81\ell^3 + 4.52\ell^2)\frac{(\log h)^{1-0.99/\ell}}{\log\log h}\right).$$

Combining this with (3.55) completes the proof in the case $h \geq h_0(\ell)$. If $h \in [2, h_0(\ell)]$, it follows that

$$\Theta(\kappa, h) \leq \exp\left(\ell^2 \sum_{p|h} p^{-\frac{1}{2}}\right) \leq \exp(C_0(\ell)).$$

Inserting this in (3.55) establishes the proof if $h \in [2, h_0(\ell)]$.

Finally, we mention the modifications in the simplest case $h = 1$. In this case, we can show that $\mathcal{B}(s_1, s_2)$ defined by (3.9) satisfies $|\mathcal{B}(s_1, s_2)| \ll 1$ for $\Re(s_1), \Re(s_2) \geq -0.99$, $\mathcal{B}(0,0) = \tilde{C}_\ell$, $\mathcal{B}^{(i_1,i_2)}(0,0) \ll 1$. Using these facts instead of Lemma 3.68 and following the above argument leads to the desired result. $\qquad\square$

The proof of Proposition has been reduced to establishing Lemma (3.2).

Proof of Proposition 3.2. Throughout this proof $\sigma_1 = \Re(s_1)$ and $\sigma_2 = \Re(s_2)$. It will also be convenient to set $a_1, a_2 \in (0, 1)$. At the end of the proof we shall choose $a_1 = a_2 = 0.99$. We begin by using Lemma (2.1) with $f_1(n) = \sigma_\ell(n, s_1 + 1)$, $f_2(n) = \sigma_\ell(n, s_2 + 1)$, $\tau_1 = s_1 + s_2 + 1$, and $\tau_2 = s_1 + s_2 + 2$ it follows from (2.26) that

$$\mathcal{B}(s_1, s_2) = \sum_{g|h} \frac{1}{g^{s_1+s_2+1}} \prod_{(p,g)=1}\left(1 - \frac{\sigma_\ell(p, s_1+1)\sigma_\ell(p, s_2+1)}{p^{s_1+s_2+2}}\right)$$

$$(3.58)$$

$$\times \prod_{p^\alpha||g}\left(\sigma_\ell(p^\alpha, s_1+1)\sigma_\ell(p^\alpha, s_2+1) - \frac{\sigma_\ell(p^{\alpha+1}, s_1+1)\sigma_\ell(p^{\alpha+1}, s_2+1)}{p^{s_1+s_2+2}}\right).$$

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We now bound this expression. By (1.16), we note that

$$1 - \frac{\sigma_\ell(p, s_1 + 1)\sigma_\ell(p, s_2 + 1)}{p^{s_1+s_2+2}} = Q(p^{-s_1-1}, p^{-s_2-1}) \tag{3.59}$$

where $Q(x, y) = (1 - x)^\ell + (1 - y)^\ell - (1 - x)^\ell(1 - y)^\ell$. By Taylor expansion

(3.60)

$$Q(x,y) = \left(1 - \ell x + \binom{\ell}{2}x^2\right) + \left(1 - \ell y + \binom{\ell}{2}y^2\right) - \left(1 - \ell x + \binom{\ell}{2}x^2\right)\left(1 - \ell y + \binom{\ell}{2}y^2\right) + O_\ell(|x|^3 + |y|^3)$$
$$= 1 - \ell^2 xy + O_\ell(|x|^2|y| + |x||y|^2 + |x|^2|y|^2 + |x|^3 + |y|^3)$$
$$= 1 - \ell^2 xy + O_\ell(|x|^3 + |y|^3),$$

since $|x|, |y| \le 1$. It follows that

$$1 - \frac{\sigma_\ell(p, s_1 + 1)\sigma_\ell(p, s_2 + 1)}{p^{s_1+s_2+2}} = 1 - \frac{\ell^2}{p^{s_1+s_2+2}} + O_\ell(p^{-3-3\sigma_1} + p^{-3-3\sigma_2})$$

(3.61)

$$= 1 - \frac{\ell^2}{p^{s_1+s_2+2}} + O_\ell(p^{-3+3\max(a_1,a_2)}) \text{ for } \sigma_1 \ge -a_1, \sigma_2 \ge -a_2.$$

By (1.17), we have that $\sigma_\ell(p^j, s+1) = \tau_\ell(p^j)H_{\ell,j}(p^{-s-1})$. For $j \ge 1$, we have by (1.17)

$$\sigma_\ell(p^\alpha, s_1 + 1)\sigma_\ell(p^\alpha, s_2 + 1) - \frac{\sigma_\ell(p^{\alpha+1}, s_1 + 1)\sigma_\ell(p^{\alpha+1}, s_2 + 1)}{p^{s_1+s_2+2}}$$

$$= \tau_\ell(p^j)^2 H_{\ell,j}(p^{-s_1-1})H_{\ell,j}(p^{-s_2-1}) - \frac{\tau_\ell(p^{j+1})^2 H_{\ell,j+1}(p^{-s_1-1})H_{\ell,j+1}(p^{-s_2-1})}{p^{s_1+s_2+2}}$$

(3.62)

$$= \tau_\ell(p^j)^2(1 + O_\ell(p^{-\sigma_1-1} + p^{-\sigma_2-1})) - \frac{\tau_\ell(p^{j+1})^2}{p^{s_1+s_2+2}}(1 + O(p^{-\sigma_1-1} + p^{-\sigma_2-1}))$$

$$= \tau_\ell(p^j)^2\left(1 + O_\ell(p^{-\sigma_1-1} + p^{-\sigma_2-1}) + O\left(\ell^2\frac{(1 + \frac{j}{\ell})^2}{(1+j)^2}p^{-\sigma_1-\sigma_2-2}\right)\right)$$

$$= \tau_\ell(p^j)^2\left(1 + O_\ell(p^{-\sigma_1-1} + p^{-\sigma_2-1})\right),$$

since $\tau_\ell(p^{j+1}) = \frac{\ell+j}{j+1}\tau_\ell(p^j)$. In addition, for every $\varepsilon > 0$, $\tau_\ell(p^j) \ll_\ell p^{j\varepsilon/2}$ and we also have the estimate

$$\mathcal{C}(p, j, s_1, s_2) \ll p^{-j(\sigma_1+\sigma_2+1-\varepsilon)} \text{ for } \sigma_1 \ge -a_1, \sigma_2 \ge -a_2. \tag{3.63}$$

Using (3.61) and the last equation, we have

$$|\mathcal{B}(s_1, s_2)| \le \sum_{g|h} \frac{1}{g^{\sigma_1+\sigma_2+1}} \prod_{(p,g)=1}\left(1 + O\left(\frac{\ell^2}{p^{\sigma_1+\sigma_2+2}} + p^{-3+a_1} + p^{-3+a_2}\right)\right) \prod_{p^\alpha||g} \tau_\ell(p^j)^2\left(1 + O_\ell(p^{-1+\max(a_1,a_2)})\right).$$

Since $\sigma_1 + \sigma_2 \ge -0.99$ the first product is absolutely convergent. It follows that

$$|\mathcal{B}(s_1, s_2)| \ll \sum_{g|h} \frac{\tau_\ell(g)^2 j(g)}{g^{\sigma_1+\sigma_2+1}} \tag{3.64}$$

where $j(g) := \prod_{p|g}(1 + Cp^{-1+\max(a_1,a_2)})$, and $C = C(\ell) > 0$. By multiplicativity, it follows that

$$|\mathcal{B}(s_1, s_2)| \ll \prod_{p^\alpha||h} \sum_{k=0}^\alpha \frac{\tau_\ell(p^k)^2 j(p^k)}{(p^k)^{\sigma_1+\sigma_2+1}} \ll \prod_{p|h}\left(1 + \frac{\ell^2}{p^{\sigma_1+\sigma_2+1}}\right) = \Theta(\sigma_1 + \sigma_2 + 1, h),$$

valid for $\sigma_1 \ge -a_1, \sigma_2 \ge -a_2$, and $\sigma_1 + \sigma_2 \ge -0.99$.

We now establish part (ii). By (3.61) it follows that there exists a prime $p_0 = p_0(\ell)$ such that if $p \geq p_0$, then $\mathcal{C}(p, 0, s_1, s_2) \neq 0$ for $\sigma_1, \sigma_2 \geq 10^{-2}$. Also, observe that by (3.59) $\mathcal{C}(p, 0, 0, 0) = 2(1 - p^{-1})^\ell - (1 - p^{-1})^{2\ell} \neq 0$. Since for each $2 \leq p \leq p_0$, $\mathcal{C}(p, 0, s_1, s_2)$ is a continuous function of s_1 and s_2, there exists $\varepsilon_0 \in (0, 10^{-2})$ such that $\mathcal{C}(p, 0, s_1, s_2) \neq 0$ for $p \in [2, p_0]$ and $|s_1| < \varepsilon_0$ and $|s_2| < \varepsilon_0$. Combining these facts, it follows that for all primes p and s_1, s_2 satisfying $|s_1| < \varepsilon_0$ and $|s_2| < \varepsilon_0$, that $\mathcal{C}(p, 0, s_1, s_2) \neq 0$. Thus we may apply Lemma (2.1) (ii). Let

$$(3.65) \qquad \mathcal{C}(p, j, s_1, s_2) = \frac{\sigma_\ell(p^j, s_1 + 1)\sigma_\ell(p^j, s_2 + 1)}{p^{j(s_1 + s_2 + 1)}} - \frac{\sigma_\ell(p^{j+1}, s_1 + 1)\sigma_\ell(p^{j+1}, s_2 + 1)}{p^{(j+1)(s_1 + s_2 + 1) + 1}}.$$

Since $\mathcal{C}(p, 0, s_1, s_2) \neq 0$ for $|s_1|, |s_2| \leq \varepsilon_0 < 10^{-2}$, Lemma (2.1) (ii) implies that

$$(3.66) \qquad \mathcal{B}(s_1, s_2) = \mathcal{B}_1(s_1, s_2)\mathcal{B}_2(s_1, s_2) \text{ for } |s_1|, |s_2| \leq \varepsilon_0,$$

where

$$(3.67) \qquad \mathcal{B}_1(s_1, s_2) = \prod_p \mathcal{C}(p, 0, s_1, s_2)$$

and

$$(3.68) \qquad \mathcal{B}_2(s_1, s_2) = \prod_{p^\alpha || h} \sum_{j=0}^\alpha \mathcal{C}(p, j, s_1, s_2)\mathcal{C}(p, 0, s_1, s_2)^{-1}.$$

We first determine the value of $\mathcal{B}(0, 0)$. It follows from (3.67), (3.59), and (3.2) that $\mathcal{B}_1(0, 0) = \tilde{C}_\ell$. Similarly, it follows from and (3.68), (3.65), (3.59), and (3.3) that $\mathcal{B}_1(0, 0) = g_\ell(h)$. Hence, $\mathcal{B}(0, 0) = \tilde{C}_\ell g_\ell(h)$.

We now establish (iii). Let ε_0 be as in part (ii). It shall be convenient to define

$$D_1 = \{s_1 \in \mathbb{C} \mid |s_1| < \varepsilon_0\} \text{ and } D_2 = \{s_2 \in \mathbb{C} \mid |s_2| < \varepsilon_0\}.$$

First observe that by the definition (3.65) and (3.61) and (3.62) we have

$$(3.69) \qquad \mathcal{C}(p, 0, s_1, s_2) = 1 - \frac{\ell^2}{p^{s_1 + s_2 + 2}} + O_\ell(p^{-2.7}) \text{ for } \sigma_1, \sigma_2 \geq -10^{-1},$$

$$(3.70) \qquad \mathcal{C}(p, j, s_1, s_2) = \frac{\tau_\ell(p^j)^2}{p^{j(s_1 + s_2 + 1)}}\left(1 + O_\ell(p^{-0.9})\right) \text{ for } \sigma_1, \sigma_2 \geq -10^{-1}.$$

From (3.67) and (3.69) we see that

$$(3.71) \qquad \mathcal{B}_1(s_1, s_2) = \prod_p \left(1 - \frac{\ell^2}{p^{s_1 + s_2 + 2}} + O_\ell(p^{-2.7})\right) \ll_\ell 1 \text{ for } \sigma_1, \sigma_2 \geq -10^{-1}.$$

By two applications of Cauchy's integral formula,

$$(3.72) \qquad \mathcal{B}_1^{(i,j)}(s_1, s_2) = O_\ell(1) \text{ for } \sigma_1, \sigma_2 \geq -10^{-2}$$

where $\mathcal{B}_1^{(i,j)}$ is defined by (1.20). We now estimate $\mathcal{B}_2(s_1, s_2)$. First, we examine each local factor at p of $\mathcal{B}_2$. Since $D_1 \times D_2 \subset \{s_1 \in \mathbb{C} \mid \sigma_1 \geq -10^{-1}\} \times \{s_2 \in \mathbb{C} \mid \sigma_2 \geq -10^{-1}\}$ it follows from (3.69) and (3.70) that

$$(3.73) \qquad \begin{aligned} \sum_{j=0}^\alpha \frac{\mathcal{C}(p, j, s_1, s_2)}{\mathcal{C}(p, 0, s_1, s_2)} &= 1 + \frac{\frac{\ell^2}{p^{s_1 + s_2 + 1}} + O(p^{-1.5})}{1 + O(p^{-1.8})} + \frac{\sum_{j=2}^\alpha p^{-0.7j}}{1 + O(p^{-1.8})} \\ &= 1 + \frac{\ell^2}{p^{s_1 + s_2 + 1}} + O(p^{-1.4}) \text{ for } (s_1, s_2) \in D_1 \times D_2. \end{aligned}$$

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Hence we can factor out a term $(1 + \frac{1}{p^{s_1+s_2+1}})^{\ell^2}$ from (3.68). Therefore we may write

(3.74) $$\mathcal{B}_2(s_1, s_2) = \Theta(s_1 + s_1 + 1, h)\mathcal{B}_3(s_1, s_2) \text{ for } (s_1, s_2) \in D_1 \times D_2,$$

where we recall that $\Theta(z, h) = \prod_{p|h}(1 + p^{-z})^{\ell^2}$ and

(3.75) $$\mathcal{B}_3(s_1, s_2) = \prod_{p^\alpha || h} \left(\sum_{j=0}^{\alpha} \mathcal{C}(p, j, s_1, s_2)\mathcal{C}(p, 0, s_1, s_2)^{-1} \right) \left(1 + \frac{1}{p^{s_1+s_2+1}}\right)^{-\ell^2}.$$

It follows from (3.73) and (3.75) that

$$\mathcal{B}_3(s_1, s_2) = \prod_{p|h}(1 + O_\ell(p^{-\sigma_1-\sigma_2-2})) \ll_\ell \prod_p (1 + O_\ell(p^{-1.8})) \ll_\ell 1 \text{ for } (s_1, s_2) \in D_1 \times D_2.$$

By Cauchy's integral formula it follows that

(3.76) $$\mathcal{B}_3^{(i_1, i_2)}(0, 0) \ll_\ell 1.$$

We also require an estimate for the partial derivatives of $\mathcal{B}$. By (3.66) and (3.74) it follows that

(3.77) $$\mathcal{B}(s_1, s_2) = \Theta(s_1 + s_1 + 1, h)\tilde{B}(s_1, s_2) \text{ for } (s_1, s_2) \in D_1 \times D_2,$$

where $\mathcal{B}(s_1, s_2) = \mathcal{B}_1(s_1, s_2)\mathcal{B}_3(s_1, s_2)$. Note that the generalized product rule, (3.72), and (3.76) imply

(3.78) $$\tilde{B}^{(i_1, i_2)}(0, 0) \ll_\ell 1.$$

By two applications of the generalized product rule to (3.77)

$$\mathcal{B}^{(i_1, i_2)}(s_1, s_2) = \sum_{a_1 + a_2 = i_1} \binom{i_1}{a_1} \sum_{a_3 + a_4 = i_2} \binom{i_2}{a_3} \left(\frac{\partial^{a_1}}{\partial s_1^{a_1}} \frac{\partial^{a_3}}{\partial s_2^{a_3}} \Theta(s_1 + s_1 + 1, h) \right) \tilde{B}^{(a_2, a_4)}(s_1, s_2).$$

Note that

(3.79) $$\frac{\partial^{a_1}}{\partial s_1^{a_1}} \frac{\partial^{a_3}}{\partial s_2^{a_3}} \Theta(s_1 + s_1 + 1, h) = \frac{d^{a_1+a_3}}{dz^{a_1+a_3}} \Theta(z, h) \Big|_{z = s_1 + s_2 + 1}.$$

By (3.76) and (3.79) it follows that

$$\mathcal{B}^{(i_1, i_2)}(0, 0) \ll_\ell \sum_{\alpha=0}^{i_1 + i_2} \Theta^{(\alpha)}(1, h)$$

We now demonstrate for $\alpha \geq 1$

(3.80) $$\Theta^{(\alpha)}(1, h) \ll \Theta(1, h)(\log\log h)^\alpha.$$

We begin by remarking that

(3.81) $$\Theta^{(1)}(z, h) = -\ell^2 \Theta(z, h)\eta(z, h)$$

where

(3.82) $$\eta(z, h) = \sum_{p|h} \frac{\log p}{p^z + 1}.$$

By the product rule it follows that

(3.83) $$\Theta^{(\alpha)}(z, h) = -\ell^2 \sum_{u_1 + u_2 = \alpha - 1} \binom{\alpha - 1}{u_1} \Theta^{(u_1)}(z, h)\eta^{(u_2)}(z, h).$$

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A calculation demonstrates that for $u \geq 0$

$$\eta^{(u)}(1, h) = \sum_{p|h} \frac{(\log p)^u}{p} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}(-j)^u}{p^{j-1}}$$

and thus

$$(3.84) \qquad \eta^{(u)}(1, h) \ll \sum_{p|h} \frac{(\log p)^u}{p} \ll \sum_{p \leq \log h} \frac{(\log p)^u}{p} + \frac{(\log \log h)^u}{\log h} \sum_{p|h} 1 \ll (\log \log h)^u.$$

We now show (3.80). The case $\alpha = 1$ follows from (3.81) and (3.84) with $u = 1$. By induction, using (3.83) and (3.84), we establish (3.80) for all $\alpha \geq 1$. From (3.80) we now have

$$(3.85) \qquad \mathcal{B}^{(i_1, i_2)}(0, 0) \ll_{\ell} \Theta(1, h)(\log \log h)^{i_1 + i_2}.$$

$\square$

Proof of Proposition 3.3. Using the first bound, we find the contribution from $|t| \leq 1$ to the integral is $O(1)$. We now treat the range $|t| \geq 1$. It is convenient to set $I(\tau, t) = \int_0^t |\zeta(\tau + iu)|^{\ell} du$. It is well known that for every $\varepsilon > 0$,

$$(3.86) \qquad I(\tau, t) \ll t^{1+\varepsilon} \text{ for } \tau \geq 1 - 1/\ell.$$

This follows from [44, Theorems 7.5,7.7]. Note that in the case of Theorem 7.7 of [44], the bound $I(\tau, t) \ll t$ for $\tau > 1 - 1/\ell$ is stated, however a minor modification of the proof yields (3.86). Since the integrand is even with respect to t the remaining range is

$$(3.87)
\begin{aligned}
2\varepsilon^{-1} \int_1^{\infty} |s(s+1)|^{-1} |\zeta(s+1)|^{\ell} dt &\ll \varepsilon^{-1} \int_1^{\infty} |\zeta(\sigma + 1 + it)|^{\ell} t^{-2} dt \\
&\ll \varepsilon^{-1} \left(-I(\sigma + 1, 1) + 2 \int_1^{\infty} I(\sigma + 1, t) t^{-3} dt \right) \\
&\ll \varepsilon^{-1} \left(1 + \int_1^{\infty} t^{-2+\varepsilon} dt \right) \ll \varepsilon^{-1}
\end{aligned}$$

by an integration by parts and (3.86).

$\square$

4. A PROBABILISTIC METHOD FOR DETERMINING MAIN TERM OF $D_k(x, h)$

In this section, we use a simple heuristic probabilistic method to rederive the conjectured formula

$$(4.1) \qquad D_k(x, h) \sim \frac{C_k f_k(h)}{(k-1)!^2} x (\log x)^{2k-2} \text{ for } 1 \leq h \leq x^{1-\varepsilon}$$

for x large, ε arbitrarily small, and recall that C_k is defined by (1.3) and $f_k(h)$ is defined by (1.4). In section three, we derived this conjecture using Duke, Friedlander, and Iwaniec's δ method, a variant of the circle method. The argument in this section has been used to derive conjectures for $\sum_{n \leq x} \Lambda(n) \Lambda(n + h)$, where $\Lambda(n)$ is the von Mangoldt function (for full details see [8] and [38]). The extension to the case of multiplicative functions was explained to the first author by Andrew Granville. We now proceed with our heuristic derivation of (4.1). It is well known that

$$(4.2) \qquad \sum_{n \leq x} \tau_k(n) \sim \frac{1}{(k-1)!} x (\log x)^{k-1}.$$

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It follows that on average $\tau_k(n)$ in the interval $[1,x]$ is $\frac{1}{(k-1)!}x(\log x)^{k-1}$. Similarly, for $1 \leq h \leq x^{1-\varepsilon}$, $\tau_k(n+h)$ in the interval $[1,x]$ is also $\frac{1}{(k-1)!}x(\log x)^{k-1}$. Thus it is reasonable to believe that for $1 \leq h \leq x^{1-\varepsilon}$, $\tau_k(n)\tau_k(n+h)$ is on average $\frac{1}{(k-1)!^2}(\log x)^{2k-2}$ in $[1,x]$. However, we must take into consideration that the values of $\tau_k(n)$ and $\tau_k(n+h)$ are not independent. For instance, if $h = p$ is prime, then if $p \nmid n$ we also have $p \nmid n + h$. The factor $C_k f_k(h)$ in (4.1) accounts for such local considerations. In order to make this precise we define a sequence of random variables $(X_p)_{p \text{ prime}}$ by

$$X_p(n) = \tau_k(p^{\mathrm{ord}_p(n)})$$

where $\mathrm{ord}_p(\cdot)$ is the p-adic valuation. Furthermore, we define

$$X_{p;h}(n) = X_p(n + h) = \tau_k(p^{\mathrm{ord}_p(n+h)}).$$

Associated to a random variable $Y : \mathbb{N} \to \mathbb{C}$ with image $\mathrm{im}(Y) = \{Y(n) \mid n \in \mathbb{N}\}$, its expected value to be

(4.3) $$\mathbb{E}(Y) = \sum_{i \in \mathrm{im}(Y)} i \cdot \mathbb{P}(Y = i)$$

where for $B \subseteq \mathbb{N}$,

(4.4) $$\mathbb{P}(B) = \lim_{X \to \infty} \frac{\#\{1 \leq n \leq X \mid n \in B\}}{X}.$$

With these definitions in hand, it is natural to make the following conjecture.

Conjecture. For $\varepsilon \in (0,1)$, x large, and $1 \leq h \leq x^{1-\varepsilon}$,

(4.5) $$\frac{1}{x}D_k(x,h) \sim \left(\prod_p \frac{\mathbb{E}(X_p X_{p;h})}{\mathbb{E}(X_p)\mathbb{E}(X_{p;h})} \right) \left(\frac{1}{x} \sum_{n \leq x} \tau_k(n) \right) \left(\frac{1}{x} \sum_{n \leq x} \tau_k(n + h) \right)$$

as $x \to \infty$.

The product in the above conjecture is the correction factor taking into account that the values of $\tau_k(n)$ and $\tau_k(n + h)$ are not independent. Each local factor in the product measures the lack of independence of X_p and $X_{p;h}$. We shall prove that the product equals $C_k f_k(h)$, which we computed earlier via the δ-method.

Proposition 4.1. *Let $k, h \in \mathbb{N}$. Then*

$$\prod_p \frac{\mathbb{E}(X_p X_{p;h})}{\mathbb{E}(X_p)\mathbb{E}(X_{p;h})} = C_k f_k(h).$$

By (4.5), Proposition 4.1, and (4.2) we have that

(4.6) $$\frac{1}{x}D_k(x,h) \sim C_k f_k(h) \log^{k-1}(x) \log^{k-1}(x + h) \sim C_k f_k(h) \log^{2k-2}(x) \text{ as } x \to \infty \text{ for } h \leq x^{1-\varepsilon}.$$

This yields the Additive Divisor Conjecture (simplified version) stated in the introduction.

The above proposition is deduced from the next lemma.

Lemma 4.2. *(i) For every prime p,*

(4.7) $$\mathbb{E}(X_p) = \mathbb{E}(X_{p;h}) = \left(1 - \frac{1}{p}\right)^{-(k-1)}.$$

(ii) If $p \nmid h$, then

(4.8) $$\mathbb{E}(X_p X_{p;h}) = -1 + 2\left(1 - \frac{1}{p}\right)^{-(k-1)}.$$

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(iii) If $p^\alpha \| h$, then

(4.9) $$\mathbb{E}(X_p X_{p;h}) = \left(1 - \frac{1}{p}\right)\left(\sum_{i=0}^{\alpha-1} \frac{\tau_k(p^i)^2}{p^i} + 2\sum_{j=0}^{\infty} \frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+j})}{p^{\alpha+j}} - \frac{\tau_k(p^\alpha)^2}{p^\alpha}\left(1 - \frac{1}{p}\right)^{-1}\right).$$

We now demonstrate the proof of Proposition 4.1 based on this lemma.

Proof of Proposition 4.1. If $p \nmid h$, then by Lemma 4.2 (i) and (ii)

$$\frac{\mathbb{E}(X_p X_{p;h})}{\mathbb{E}(X_p)\mathbb{E}(X_{p;h})} = \frac{-1 + 2\left(1 - \frac{1}{p}\right)^{-(k-1)}}{\left(1 - \frac{1}{p}\right)^{-2(k-1)}} = 2\left(1 - \frac{1}{p}\right)^{k-1} - \left(1 - \frac{1}{p}\right)^{2(k-1)}.$$

Therefore

(4.10)
$$\prod_p \frac{\mathbb{E}(X_p X_{p;h})}{\mathbb{E}(X_p)\mathbb{E}(X_{p;h})} = C_k \prod_{p^\alpha \| h} \frac{\mathbb{E}(X_p X_{p;h})}{\left(2\left(1 - \frac{1}{p}\right)^{-(k-1)} - 1\right)}$$

$$= C_k \prod_{p^\alpha \| h} \frac{\left(1 - \frac{1}{p}\right)\left(\sum_{i=0}^{\alpha-1} \frac{\tau_k(p^i)^2}{p^i} + 2\sum_{j=0}^{\infty} \frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+j})}{p^{\alpha+j}} - \frac{\tau_k(p^\alpha)^2}{p^\alpha}\left(1 - \frac{1}{p}\right)^{-1}\right)}{\left(2\left(1 - \frac{1}{p}\right)^{-(k-1)} - 1\right)},$$

by an application of Lemma 4.2 part (iii). In order to finish our argument, we must establish that

(4.11) $$f_k(p^\alpha) = \frac{\left(1 - \frac{1}{p}\right)\left(\sum_{i=0}^{\alpha-1} \frac{\tau_k(p^i)^2}{p^i} + 2\sum_{j=0}^{\infty} \frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+j})}{p^{\alpha+j}} - \frac{\tau_k(p^\alpha)^2}{p^\alpha}\left(1 - \frac{1}{p}\right)^{-1}\right)}{\left(2\left(1 - \frac{1}{p}\right)^{-(k-1)} - 1\right)}.$$

By (1.19) it suffices to prove
(4.12)
$$\sum_{j=0}^{\alpha} \left(\frac{\sigma_{k-1}(p^j, 1)^2}{p^j} - \frac{\sigma_{k-1}(p^{j+1}, 1)^2}{p^{j+2}}\right) = \left(1 - \frac{1}{p}\right)^{2k-1}\left(\sum_{i=0}^{\alpha-1} \frac{\tau_k(p^i)^2}{p^i} + 2\sum_{j=0}^{\infty} \frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+j})}{p^{\alpha+j}} - \frac{\tau_k(p^\alpha)^2}{p^\alpha}\left(1 - \frac{1}{p}\right)^{-1}\right).$$

Let N_{p^α} denote the left hand side of (4.12). By a difference of squares,

$$N_{p^\alpha} = \sum_{j=0}^{\alpha} \frac{1}{p^j}\left(\sigma_{k-1}(p^j, 1) - \frac{\sigma_{k-1}(p^{j+1}, 1)}{p}\right)\left(\sigma_{k-1}(p^j, 1) + \frac{\sigma_{k-1}(p^{j+1}, 1)}{p}\right)$$

and from the definition (1.15)

$$\sigma_{k-1}(p^j, 1) - \frac{\sigma_{k-1}(p^{j+1}, 1)}{p} = \left(1 - \frac{1}{p}\right)^{k-1}\left(\sum_{i=0}^{\infty} \frac{\tau_{k-1}(p^{j+i})}{p^i} - \sum_{i=0}^{\infty} \frac{\tau_{k-1}(p^{j+1+i})}{p^{i+1}}\right),$$

$$\sigma_{k-1}(p^j, 1) + \frac{\sigma_{k-1}(p^{j+1}, 1)}{p} = \left(1 - \frac{1}{p}\right)^{k-1}\left(\sum_{i=0}^{\infty} \frac{\tau_{k-1}(p^{j+i})}{p^i} + \sum_{i=0}^{\infty} \frac{\tau_{k-1}(p^{j+1+i})}{p^{i+1}}\right).$$

Combining the above equations and simplifying further

$$N_{p^\alpha} = \left(1 - \frac{1}{p}\right)^{2k-2} \sum_{j=0}^{\alpha} \frac{\tau_{k-1}(p^j)}{p^j}\left(2\sum_{i=0}^{\infty} \frac{\tau_{k-1}(p^{j+i})}{p^i} - \tau_{k-1}(p^j)\right).$$

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If $j = 0$, then by (1.12) the term in the above brackets is $2(1 - p^{-1})^{-(k-1)} - 1$. If $j \geq 1$, then we use (1.13) which implies $\tau_{k-1}(p^{j+i}) = \tau_k(p^{j+i}) - \tau_k(p^{j+i-1})$. Thus

$$\sum_{i=0}^{\infty} \frac{\tau_{k-1}(p^{j+i})}{p^i} = \sum_{i=0}^{\infty} \frac{\tau_k(p^{j+i})}{p^i} - \sum_{i=0}^{\infty} \frac{\tau_k(p^{j+i-1})}{p^i} = \left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^{j+i})}{p^i} - \tau_k(p^{j-1}).$$

Therefore

$$(4.13) \qquad N_{p^\alpha} = \left(1 - \frac{1}{p}\right)^{2(k-1)} (2(1 - p^{-1})^{-(k-1)} - 1 + N'_{p^\alpha})$$

where

$$(4.14) \qquad N'_{p^\alpha} = \sum_{j=1}^{\alpha} \frac{\tau_{k-1}(p^j)}{p^j} \left(2\left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^{j+i})}{p^i} - 2\tau_k(p^{j-1}) - \tau_{k-1}(p^j)\right).$$

We now simplify N'_{p^α}. Using $\tau_{k-1}(p^j) = \tau_k(p^j) - \tau_k(p^{j-1})$ twice, splitting into two sums, and then making the variable change $j \to j + 1$ in the second sum yields

$$N'_{p^\alpha} = \sum_{j=1}^{\alpha} \frac{\tau_k(p^j)}{p^j} \left(2\left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^{j+i})}{p^i} - \tau_k(p^{j-1}) - \tau_k(p^j)\right)$$

$$- \sum_{j=0}^{\alpha-1} \frac{\tau_k(p^j)}{p^{j+1}} \left(2\left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^{j+i+1})}{p^i} - \tau_k(p^j) - \tau_k(p^{j+1})\right).$$

Separating off the $j = \alpha$ term in the first sum and the $j = 0$ term in the second sum and then grouping together the $j = 1$ to $j = \alpha - 1$ terms gives

$$N'_{p^\alpha} = \frac{\tau_k(p^\alpha)}{p^\alpha} \left(2\left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^{\alpha+i})}{p^i} - \tau_k(p^{\alpha-1}) - \tau_k(p^\alpha)\right) - \frac{1}{p} \left(2\left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^{i+1})}{p^i} - \tau_k(1) - \tau_k(p)\right)$$

$$+ \sum_{j=1}^{\alpha-1} \frac{\tau_k(p^j)}{p^j} \left(2\left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^{j+i})}{p^i} - \tau_k(p^{j-1}) - \tau_k(p^j) - \left(2\left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^{j+i+1})}{p^{i+1}} - \frac{\tau_k(p^j)}{p} - \frac{\tau_k(p^{j+1})}{p}\right)\right)$$

$$= \tau_k(p^\alpha) \left(2\left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^{\alpha+i})}{p^{\alpha+i}} - \frac{\tau_k(p^{\alpha-1})}{p^\alpha} - \frac{\tau_k(p^\alpha)}{p^\alpha}\right) - \left(2\left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^{i+1})}{p^{i+1}} - \frac{1}{p} - \frac{\tau_k(p)}{p}\right)$$

$$+ \sum_{j=1}^{\alpha-1} \frac{\tau_k(p^j)}{p^j} \left(2\left(1 - \frac{1}{p}\right) \tau_k(p^j) - \tau_k(p^{j-1}) - \tau_k(p^j) + \frac{\tau_k(p^j)}{p} + \frac{\tau_k(p^{j+1})}{p}\right)\right).$$

By (1.12) we have $\sum_{i=0}^{\infty} \frac{\tau_k(p^{i+1})}{p^{i+1}} = (1 - p^{-1})^{-k} - 1$, and thus

$$N'_{p^\alpha} = 2\left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+i})}{p^{\alpha+i}} - \frac{\tau_k(p^\alpha)\tau_k(p^{\alpha-1})}{p^\alpha} - \frac{\tau_k(p^\alpha)^2}{p^\alpha} - 2\left(1 - \frac{1}{p}\right)^{-(k-1)} + 2\left(1 - \frac{1}{p}\right) + \frac{1}{p} + \frac{\tau_k(p)}{p}$$

$$+ \left(1 - \frac{1}{p}\right) \sum_{j=1}^{\alpha-1} \frac{\tau_k(p^j)^2}{p^j} - \frac{\tau_k(p)}{p} + \frac{\tau_k(p^{\alpha-1})\tau_k(p^\alpha)}{p^\alpha}$$

$$= 2\left(1 - \frac{1}{p}\right) \sum_{i=0}^{\infty} \frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+i})}{p^{\alpha+i}} - \frac{\tau_k(p^\alpha)^2}{p^\alpha} - 2\left(1 - \frac{1}{p}\right)^{-(k-1)} + 2 - \frac{1}{p} + \left(1 - \frac{1}{p}\right) \sum_{j=1}^{\alpha-1} \frac{\tau_k(p^j)^2}{p^j}.$$

It follows that

$$N_{p^\alpha} = \left(1 - \frac{1}{p}\right)^{2k-2}\left(2(1-p^{-1})^{-(k-1)} - 1 + 2\left(1 - \frac{1}{p}\right)\sum_{i=0}^{\infty}\frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+i})}{p^{\alpha+i}}\right.$$

$$\left. - \frac{\tau_k(p^\alpha)^2}{p^\alpha} - 2\left(1 - \frac{1}{p}\right)^{-(k-1)} + 2 - \frac{1}{p} + \left(1 - \frac{1}{p}\right)\sum_{j=1}^{\alpha-1}\frac{\tau_k(p^j)^2}{p^j}\right)$$

$$= \left(1 - \frac{1}{p}\right)^{2k-2}\left(1 - \frac{1}{p} + 2\left(1 - \frac{1}{p}\right)\sum_{i=0}^{\infty}\frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+i})}{p^{\alpha+i}} - \frac{\tau_k(p^\alpha)^2}{p^\alpha} + \left(1 - \frac{1}{p}\right)\sum_{j=1}^{\alpha-1}\frac{\tau_k(p^j)^2}{p^j}\right)$$

$$= \left(1 - \frac{1}{p}\right)^{2k-1}\left(\sum_{j=0}^{\alpha-1}\frac{\tau_k(p^j)^2}{p^j} + 2\sum_{i=0}^{\infty}\frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+i})}{p^{\alpha+i}} - \frac{\tau_k(p^\alpha)^2}{p^\alpha}\left(1 - \frac{1}{p}\right)^{-1}\right).$$

Substituting this last expression in (4.13) completes the proof of (4.12). □

Before establishing Lemma 4.2, we make a few observations.

$$(4.15) \qquad\qquad \mathbb{P}\Big(\{n \in \mathbb{N} \mid p \nmid n\}\Big) = 1 - \frac{1}{p}$$

and for $i \geq 1$,

$$(4.16) \qquad\qquad \mathbb{P}\Big(\{n \in \mathbb{N} \mid p^i \| n\}\Big) = \frac{1}{p^i} - \frac{1}{p^{i+1}},$$

where $\mathbb{P}$ is defined by (4.4). Idenitity (4.15) is since n lies in $p - 1$ of p residue classes modulo p and (4.16) follows from writing $n = p^i n'$ where $(n', p) = 1$.

Proof of Lemma 4.2. (i) First, we compute $\mathbb{E}(X_p)$. The values of X_p are precisely $\tau_k(p^i)$ for $i \geq 0$. Note that if $i = 0$, then $X_p = 1$. This means that $p \nmid n$ and the probability of this occurring is $1 - \frac{1}{p}$, since n lies in $p - 1$ of p residue classes modulo p. Now $X_p(n) = \tau_k(p^i)$ with $i \geq 1$, precisely when $p^i \| n$. This occurs with probability $\frac{1}{p^i} - \frac{1}{p^{i+1}}$. Therefore

$$\mathbb{E}(X_p) = 1 - \frac{1}{p} + \sum_{i=0}^{\infty}\tau_k(p^i)\left(\frac{1}{p^i} - \frac{1}{p^{i+1}}\right) = \left(1 - \frac{1}{p}\right)\sum_{i=0}^{\infty}\frac{\tau_k(p^i)}{p^i} = \left(1 - \frac{1}{p}\right)^{-(k-1)}.$$

A similar argument establishes $\mathbb{E}(X_{p;h}) = (1 - \frac{1}{p})^{-(k-1)}$.

(ii) We now compute $\mathbb{E}(X_p X_{p;h})$, in the case $p \nmid h$. If $n \not\equiv 0, -h \pmod{p}$, then $X_p(n) = X_{p;h}(n) = 1$. The probability of this occurring is $\frac{p-2}{p} = 1 - \frac{2}{p}$. If $n \equiv 0 \pmod{p}$ and $p^i \| n$ with $i \geq 1$, then $p \nmid n+h$. Therefore $X_p(n) = \tau_k(p^i)$ and $X_{p;h}(n) = 1$. The probability of this occurring is $\frac{1}{p^i} - \frac{1}{p^{i+1}}$. Similarly, if $n \equiv -h \pmod{p}$ and $p^i \| n + h$ with $i \geq 1$, then $p \nmid n$. Therefore $X_p(n) = 1$ and $X_{p;h}(n) = \tau_k(p^i)$ and the probability of this occurring is $\frac{1}{p^i} - \frac{1}{p^{i+1}}$. It follows that

$$\mathbb{E}(X_p X_{p;h}) = 1 - \frac{2}{p} + 2\sum_{i=1}^{\infty}\tau_k(p^i)\left(\frac{1}{p^i} - \frac{1}{p^{i+1}}\right).$$

Now

$$\sum_{i=1}^{\infty}\tau_k(p^i)\left(\frac{1}{p^i} - \frac{1}{p^{i+1}}\right) = \left(1 - \frac{1}{p}\right)\sum_{i=1}^{\infty}\frac{\tau_k(p^i)}{p^i} = \left(1 - \frac{1}{p}\right)\left(\left(1 - \frac{1}{p}\right)^{-k} - 1\right)$$

$$= \left(1 - \frac{1}{p}\right)^{-(k-1)} - 1 + \frac{1}{p}$$

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and thus $\mathbb{E}(X_p X_{p;h}) = -1 + 2(1 - \frac{1}{p})^{-(k-1)}$.

(iii)We now compute $\mathbb{E}(X_p X_{p;h})$, in the case $p^\alpha \,||\, h$.

If $n \not\equiv 0 \pmod{p}$, then $n + h \not\equiv 0 \pmod{p}$. This is since if $p \mid n + h$, then $p \mid n$ as $p^\alpha \,||\, h$. This case occurs with probability $1 - \frac{1}{p}$ and for these n, $X_p(n) = X_{p;h}(n) = 1$. These terms contribute

$$(4.17) \qquad 1 \cdot \left(1 - \frac{1}{p}\right) = 1 - \frac{1}{p}$$

to $\mathbb{E}(X_p X_{p;h})$. Now consider $p^i \,||\, n$ with $i \geq 1$. In this case, $X_p(n) = \tau_k(p^i)$. We now determine the order of p dividing $n + h$. Writing $n = p^i n'$ and $h = p^\alpha h'$ with $(n', p) = (h', p) = 1$, we have

$$n + h = p^i n' + p^\alpha h' = p^{\min(i,\alpha)}(n' p^{i - \min(i,\alpha)} + h' p^{\alpha - \min(i,\alpha)}).$$

Note that if $i \neq \alpha$, then $\operatorname{ord}_p(n+h) = \min(i, \alpha)$ and $X_{p;h}(n) = \tau_k(p^{\min(i,\alpha)})$. These terms make a contribution

$$(4.18) \qquad \sum_{\substack{i=1 \\ i \neq \alpha}}^\infty \tau_k(p^i)\tau_k(p^{\min(i,\alpha)})\left(\frac{1}{p^i} - \frac{1}{p^{i+1}}\right).$$

to $\mathbb{E}(X_p X_{p;h})$. However, if $i = \alpha$, then $X_p(n) = \tau_k(p^\alpha)$. Now we determine the power of p dividing $n + h$. Since $n + h = p^\alpha(n' + h')$, the p-adic valuation depends on the order of p dividing $n' + h'$. Since $(n', p) = 1$, it falls in $p - 1$ residue classes modulo p. If $n' \not\equiv -h' \pmod{p}$, then $\operatorname{ord}_p(n + h) = \alpha$. If $n' \equiv -h' \pmod{p}$, then there exists $j \geq 1$ such that $p^j \,||\, n' + h'$ and $\operatorname{ord}_p(n + h) = \alpha + j$. By these observations we have the disjoint union

$$\{n \in \mathbb{N} \mid p^\alpha \,||\, n\} = A_0 \cup \bigcup_{j=1}^\infty A_j$$

where $A_0 = \{n \in \mathbb{N} \mid p^\alpha \,||\, n, n' \not\equiv -h' \pmod{p}\}$, $A_j = \{n \in \mathbb{N} \mid p^\alpha \,||\, n, p^j \,||\, n' + h'\}$, and $n' = \frac{n}{p^\alpha}$. Since $(n', p) = 1$ and $n' \not\equiv -h' \pmod{p}$, it follows that $\mathbb{P}(A_0) = \frac{1}{p^\alpha}(1 - \frac{2}{p})$ as n' lies in $p - 2$ residue classes modulo p. A similar calculation establishes that $\mathbb{P}(A_j) = \frac{1}{p^\alpha}(\frac{1}{p^j} - \frac{1}{p^{j+1}})$. If $n \in A_0$, then $X_{p;h}(n) = \tau_k(p^\alpha)$ and if $n \in A_j$, then $X_{p,h}(n) = \tau_k(p^{\alpha+j})$. The contribution from all terms with $p^\alpha \,||\, n$ is

$$(4.19) \qquad \frac{\tau_k(p^\alpha)^2}{p^\alpha}\left(1 - \frac{2}{p}\right) + \sum_{j=1}^\infty \frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+j})}{p^\alpha}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right).$$

Combining (4.17), (4.18), and (4.19) yields

$$\mathbb{E}(X_p X_{p;h}) = 1 - \frac{1}{p} + \left(1 - \frac{1}{p}\right)\sum_{\substack{i \geq 1 \\ i \neq \alpha}} \frac{\tau_k(p^i)\tau_k(p^{\min(i,\alpha)})}{p^i} + \frac{\tau_k(p^\alpha)^2}{p^\alpha}\left(1 - \frac{2}{p}\right) + \sum_{j=1}^\infty \frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+j})}{p^\alpha}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)$$

$$= \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\substack{i \geq 1 \\ i \neq \alpha}} \frac{\tau_k(p^i)\tau_k(p^{\min(i,\alpha)})}{p^i} + \frac{\tau_k(p^\alpha)^2}{p^\alpha}\left(1 - \frac{2}{p}\right)\left(1 - \frac{1}{p}\right)^{-1} + \sum_{j=1}^\infty \frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+j})}{p^{\alpha+j}}\right)$$

$$= \left(1 - \frac{1}{p}\right)\left(\sum_{i=0}^{\alpha-1} \frac{\tau_i(p^i)^2}{p^i} + 2\sum_{j=0}^\infty \frac{\tau_k(p^\alpha)\tau_k(p^{\alpha+j})}{p^{\alpha+j}} - \frac{\tau_k(p^\alpha)^2}{p^\alpha}\left(1 - \frac{1}{p}\right)^{-1}\right).$$

This establishes (4.9) and completes the proof of Lemma 4.2. $\qquad\qquad \square$

5. Concluding remarks

In this article, we studied the sum $D_k(x, h)$. Lower bounds for this sum were obtained and the main term in its conjectured asymptotic was studied. We now mention several avenues of possible future research.

(1) Improve the lower bounds for $D_k(x, h)$. One might attempt to use inequalities of the shape

$$\tau_k(n) \geq \sum_{\substack{\alpha_1 \cdots \alpha_k = n \\ \prod_{i \in \mathcal{J}} \alpha_i \leq x^{\beta_i}}} 1$$

where $\mathcal{J}$ ranges over certain subsets of $\{1, \ldots, k\}$ and $\sum_{i=1}^{k} \beta_i \leq 1/2$.

(2) Establish a version of the uniform bound (1.9), making the k dependence explicit. Currently, even the bound (1.9) for $h \leq x^C$ does not appear in a published reference.

(3) It seems possible that the probabilistic method of section 4 can be used to obtain the full main term asymptotic for $D_k(x, h)$.

(4) Study the more general sums

$$(5.1) \qquad \sum_{n \leq x} \tau_{k_1}(n + h_1) \cdots \tau_{k_r}(n + h_r)$$

where $r \geq 2$, $k_1, \ldots, k_r \in \mathbb{R}^+$ and $h_1, \ldots, h_r \in \mathbb{Z}$. It is likely that the methods of this article may be applied to obtain lower bounds for (5.1) of the correct order of magnitude and to write down conjectural asymptotic formula for this sum. The asymptotic evaluation of (5.1) is an open problem and this is well-known to the experts. [3] For instance, it is an open problem to evaluate the sum

$$\sum_{n \leq x} \tau(n)\tau(n+1)\tau(n+2).$$

It should be noted that Blomer [4] recently succeeded in evaluating the triple correlation sum

$$\sum_{x \leq n \leq 2x} \tau(n-h)\tau(n)\tau(n+h)$$

on average over h.

(5) Study the lower order terms in the main term asymptotic for $D_k(x, h)$. More precisely, determine explicit expressions for the coefficients $\alpha_i(h)$ for $0 \leq i \leq 2k - 3$ and numerically study the size of $|D_k(x, h) - P_{2k-2;h}(\log x)|$ with h as a function of x. This might provide evidence towards the true sizes of the constants θ_k and β_k in Conjecture (2.2). Furthermore, it seems possible to use the probabilistic method of section 5 to calculate the lower order terms.

References

[1] J. C. Andrade, L. Bary-Soroker, Z. Rudnick, *Shifted convolution and the Titchmarsh divisor problem over $F_q[t]$*, Number fields and function fields: coalescences, contrasts and emerging applications. Phil. Trans. R. Soc. A 373: 20140315. (2015).

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[3] The first author gave a talk at ICERM (Brown University) on Nov. 12, 2015 and mentioned this open problem.

[2] F. Aryan, *Binary and quadratic divisor problems*, to appear in International Journal of Number Theory, preprint available at `arxiv.org/abs/1502.06067`.

[3] S. Baier, T. Browning, G. Marasingha, and L. Zhao, *Averages of shifted convolutions of $d_3(n)$*, Proc. Edinb. Math. Soc. (2) **55** (2012), no. 3, 551-576.

[4] V. Blomer, *On triple correlations of divisor functions*, preprint, `arxiv.org/abs/1512.03278`.

[5] R.P. Brent, *Irregularities in the Distribution of Primes and Twin Primes*, Comp. Math. **29** (1975), no. 129, 43-56.

[6] A. Bykovskiĭ and A. I. Vinogradov, *Inhomogeneous convolutions* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **160** (1987), no. Anal. Teor. Chisel i Teor.Funktsii. 8, 16 30, 296; translation in J. Soviet Math. **52** (1990), no. 3, 3004-3016.

[7] V. Chandee and X. Li, *Lower bounds for small fractional moments of Dirichlet L-functions*, Int. Math. Res. Not. 2013, no. 19, 4349-4381.

[8] Cherwell, *Note on the distribution of the intervals between prime numbers*, Quart. J. Math. Oxford **17** (1946), no. 65, 46-62.

[9] J.B. Conrey, S.M. Gonek, *High moments of the Riemann zeta-function*, Duke Math. J. **107** (2001), 577-604.

[10] J.B. Conrey and J.P. Keating, *Moments of zeta and correlations of divisor-sums: III*, Indagationes Mathematicae, vol 26., pp. 736-747.

[11] Stephen Daniel, *Uniform bounds for short sums of certain arithmetic functions of polynomial arguments*, unpublished manuscript.

[12] J.-M. Deshouillers, *Majorations en moyenne de sommes de Kloosterman*, Seminar on Number Theory, 1981/1982, Exp. No. 3, 5 pp, Univ. Bordeaux I, Talence, 1982.

[13] J.-M. Deshouillers, H. Iwaniec, *An additive divisor problem*, J. London Math. Soc. **26** (1982), 1-14.

[14] S. Drappeau, *Sums of Kloosterman sums in arithmetic progressions, and the error term in the dispersion method*, preprint, `arxiv.org/abs/1504.05549`.

[15] F. Dress, H. Iwaniec, and G. Tenenbaum, *Sur une somme liée à la fonction de Möbius*, J. Reine Angew. Math. **340** (1983), 53-58.

[16] W. Duke, J.B. Friedlander, and H. Iwaniec, *A quadratic divisor problem*, Invent. Math. **115** (1994), no. 2, 209-217.

[17] T. Estermann, *Über die Darstellung einer Zahl als Differenz von zwei Produkten*, J. Reine Angew. Math. **164** (1931), 173-182.

[18] J. Friedlander, H. Iwaniec, *A polynomial divisor problem*, J. Reine Angew. Math. **601** (2006), 109-137.

[19] J. Friedlander, H. Iwaniec, *Divisor weighted sums*, Journal of Mathematical Sciences **137** (2006), no. 2, 4739-4743.

[20] É. Fouvry and G. Tenenbaum, *Sur la corrélation des fonctions de Piltz*, Rev. Mat. Iberoamericana **1** (1985), no. 3, 43-54.

[21] D. Goldston, J. Pintz, C. Yildirim, *Primes in tuples. I*, Ann. of Math. (2) **170** (2009), no. 2, 819-862.

[22] D.R Heath-Brown, *The fourth power moment of the Riemann zeta-function*, Proc. London Math. Soc. **3** (1979), 385-422.

[23] K. Henriot, *Nair-Tenenbaum bounds uniform with respect to the discriminant*, Math Proceedings of the Cambridge Philosophical Society **153** (2012), no. 3, 405-424.

[24] K. Henriot, *Nair-Tenenbaum bounds uniform with respect to the discriminant - erratum*, Math Proceedings of the Cambridge Philosophical Society.

[25] K. Henriot, personal communication.

[26] A.E. Ingham, *Mean-value theorems in the theory of the Riemann zeta function*, P.L.M.S. (2) **27** (1926), 273-300.

[27] A. Ingham, *Some asymptotic formulae in the theory of numbers*, J. London Math. Soc. **2** (1927), 202-208.

[28] A. Ivić, *The general additive divisor problem and moments of the zeta-function*, New trends in proba- bility and statistics, Vol. 4 (Palanga, 1996), 69-89, VSP, Utrecht, 1997.

[29] A. Ivić, *On the ternary additive divisor problem and the sixth moment of the zeta-function*, Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), 205-243, London Math. Soc. Lecture Note Ser., 237, Cambridge Univ. Press, Cambridge, 1997.

[30] A. Ivić, J. Wu, *On the general additive divisor problem*, (English summary) Tr. Mat. Inst. Steklova 276 (2012), Teoriya Chisel, Algebra i Analiz, 146-154; translation in Proc. Steklov Inst. Math. 276 (2012), no. 1, 140-148.

[31] O. Klurman, *Correlations of multiplicative functions and applications*, preprint, `arxiv.org/abs/1603.08453`.

[32] Y. Linnik, *The dispersion method in binary additive problems*, American Math. Soc., Providence, Rhode Island, 1963.

[33] T. Meurman, *On the binary additive divisor problem*, Number theory (Turku, 1999), 223-246, de Gruyter, Berlin, 2001.

[34] Y. Motohashi, *On some additive divisor problems*, J. Math. Soc. Japan **28** (1976), 772-784.

[35] Y. Motohashi, *The binary additive divisor problem*, Ann. Sci. École Norm. Sup. **27** (1994), 529-572.

[36] M. Nair and G. Tenenbaum, *Short sums of certain arithmetic functions*, Acta Math. **180** (1998), no. 1, 119-144.

[37] N. Ng, *Large gaps between the zeros of the Riemann zeta function*, J. Number Theory **128** (2008), 509–556.

[38] G. Polya, *Heuristic Reasoning in the Theory of Numbers*, Amer. Math. Monthly **66** (1959), no. 5, 375-384.

[39] G. Robin, *Estimation de la fonction de Tchebychef θ sur le k-ième nombre premier et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n*, Acta Arith. **42** (1983), no. 4, 367-389.

[40] J. B. Rosser and L. Schoenfeld, *Approximate Formulas for Some Functions of Prime Numbers*, Illinois Journal Math. **6** (1962) pp. 64-94.

[41] A. Selberg, *On an elementary method in the theory of primes*, Norske Vid. Selsk. Forh., Trondhjem **19** (1947). no. 18, 64-67.

[42] P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. Reine Angew. Math. **313** (1980), 161-170.

[43] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Stud. Adv. Math. **46**, Cambridge Univ. Press, Cambridge, 1995.

[44] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., Oxford Univ. Press, New York, 1986.

[45] E.C. Titchmarsh, *Some problems in the analytic theory of numbers*, Quart. J. Math., Oxford Ser. **13**, (1942). 129-152.

[46] B. Topacogullari, *Shifted convolution of divisor sums*, preprint, `arxiv.org/abs/1506.02608`.

[47] B. Topacogullari, *On a certain additive divisor sum*, preprint, `arxiv.org/abs/1512.05770`.

[48] B. Topacogullari, *The shifted convolution of generalized divisor functions*, preprint, `http://arxiv.org/abs/1605.02364`.

[49] A.I. Vinogradov, *Sl_n techniques and the density conjecture* (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **168** (1988), Anal. Teor. Chisel i Teor. Funktsii. 9, 5-10, 187; translation in J. Soviet Math. **53** (1991), no. 3, 225-228.

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