A NOTE ON A CONJECTURE OF GONEK

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Abstract. We derive a lower bound for a second moment of the reciprocal of the derivative of the Riemann zeta-function over the zeros of $\zeta(s)$ that is half the size of the conjectured value. Our result is conditional upon the assumption of the Riemann Hypothesis and the conjecture that the zeros of the zeta-function are simple.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. Using a heuristic method similar to Montgomery’s study [13] of the pair-correlation of the imaginary parts of the non-trivial zeros of $\zeta(s)$, Gonek has made the following conjecture [7, 8].

Conjecture. Assume the Riemann Hypothesis and that the zeros of $\zeta(s)$ are simple. Then, as $T \to \infty$,

\[
\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2} \sim \frac{3}{\pi^3} T
\]

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

The assumption on the simplicity of the zeros of the zeta-function in the above conjecture is so that the sum over zeros on the right-hand side of (1.1) is well defined. While the details of Gonek’s method have never been published, he announced his conjecture in [5]. More recently, a different heuristic method of Hughes, Keating, and O’Connell [10] based upon modeling the Riemann zeta-function and its derivative using the characteristic polynomials of random matrices has led to the same conjecture. Through the work of Ingham [11], Titchmarsh (Chapter 14 of [21]), Odlyzko and te Riele [17], Gonek (unpublished), and Ng [15], it is known that the behavior of this and related sums are intimately connected to the distribution of the

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summatory function

\[ M(x) = \sum_{n \leq x} \mu(n) \]

where \( \mu(\cdot) \), the Mőbius function, is defined by \( \mu(1) = 1 \), \( \mu(n) = (-1)^k \) if \( n \) is divisible by \( k \) distinct primes, and \( \mu(n) = 0 \) if \( n > 1 \) is not square-free. See also [9] and [20] for connections between similar sums and other arithmetic problems.

In support of his conjecture, Gonek [5] has shown, assuming the Riemann Hypothesis and the simplicity of the zeros of \( \zeta(s) \), that

\[ \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2} \geq CT \]

for some constant \( C > 0 \) and \( T \) sufficiently large. In this note, we show that the inequality in (1.2) holds for any constant \( C < \frac{3}{2\pi} \).

**Theorem.** Assume the Riemann Hypothesis and that the zeros of \( \zeta(s) \) are simple. Then, for any fixed \( \varepsilon > 0 \),

\[ \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2} \geq \left( \frac{3}{2\pi^3} - \varepsilon \right) T \]

for \( T \) sufficiently large.

While our result differs from the conjectural lower bound by a factor of 2, any improvements in the strength of this lower bound have, thus far, eluded us. It would be interesting to investigate whether for \( k > 0 \) there is a constant \( C_k > 0 \) such that

\[ \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^{2k}} \geq C_k T (\log T)^{(k-1)^2} \]

for \( T \) sufficiently large. However, a lower bound of this form is probably not of the correct order of magnitude for all \( k \). This is because it is expected that for each \( \varepsilon > 0 \) there are infinitely many zeros \( \rho = \frac{1}{2} + i\gamma \) of \( \zeta(s) \) satisfying \( |\zeta'(\rho)|^{-1} \gg |\gamma|^{1/3-\varepsilon} \). If such a sequence were to exist, it would then follow that

\[ \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^{2k}} = \Omega \left( T^{2k/3-\varepsilon} \right) \]

and the lower bound in (1.4) would be significantly weaker than this \( \Omega \)-result when \( k > \frac{3}{2} \).
2. Proof of Theorem

The method we use to prove our theorem is based on a recent idea of Rudnick and Soundararajan [18]. Let
\begin{equation}
\xi = T^\vartheta
\end{equation}
where \(0 < \vartheta < 1\) is fixed and define the Dirichlet polynomial
\[\mathcal{M}_\xi(s) = \sum_{n \leq \xi} \mu(n)n^{-s}\]
where \(\mu\) is the Möbius function. Assuming the Riemann Hypothesis, for any non-trivial zero \(\rho = \frac{1}{2} + i\gamma\) of \(\zeta(s)\), we see that \(\mathcal{M}_\xi(\rho) = \mathcal{M}_\xi(1 - \rho)\). From this observation and Cauchy’s inequality it follows that
\begin{equation}
\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2} \geq \frac{|M_1|^2}{M_2}
\end{equation}
where
\[M_1 = \sum_{0 < \gamma \leq T} \frac{1}{\zeta'(\rho)} \mathcal{M}_\xi(1 - \rho) \quad \text{and} \quad M_2 = \sum_{0 < \gamma \leq T} |\mathcal{M}_\xi(\rho)|^2.
\]
Our Theorem is a consequence of the following proposition.

**Proposition.** Assume the Riemann Hypothesis and let \(0 < \vartheta < 1\) be fixed. Then
\begin{equation}
M_2 = \frac{3}{\pi^3} (\vartheta + \vartheta^2) T \log^2 T + O(T \log T).
\end{equation}
If we further assume that the zeros of \(\zeta(s)\) are all simple, then there exists a sequence \(T := \{\tau_n\}_{n=3}^\infty\) such that \(n < \tau_n \leq n + 1\) and for \(T \in T\) we have
\begin{equation}
M_1 = \frac{3\vartheta}{\pi^3} T \log T + O(T).
\end{equation}

We now deduce our theorem from the above proposition.

**Proof of the Theorem.** Let \(T \geq 4\) and choose \(\tau_n\) to satisfy \(T - 1 \leq \tau_n < T\). Combining (2.2), (2.4), and (2.3) we see that
\begin{equation}
\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2} \geq \sum_{0 < \gamma \leq \tau_n} \frac{1}{|\zeta'(\rho)|^2} \geq \frac{\vartheta^2}{(\vartheta + \vartheta^2)} \frac{3}{\pi^3} \tau_n + o(\tau_n)
\end{equation}
\[\geq \frac{1}{(1 + \vartheta - 1)} \frac{3}{\pi^3} T + o(T)
\]
under the assumption of the Riemann Hypothesis and the simplicity of the zeros of \(\zeta(s)\). From (2.5), our theorem follows by letting \(\vartheta \to 1^-\). \(\square\)
We could have just as easily estimated the sums $M_1$ and $M_2$ using a Dirichlet polynomial $\sum_{n \leq \xi} a_n n^{-s}$ for a large class of coefficients $a_n$ in place of $M_\xi(s)$. In the special case where
\[
a_n = \mu(n) P\left(\frac{\log \xi/n}{\log \xi}\right)
\]
for polynomials $P$, we can show that the choice $P = 1$ is optimal in the sense that it leads to largest lower bound in (1.3).

We prove the above proposition in the next two sections; the sum $M_1$ is estimated in section 3 and the sum $M_2$ is estimated in section 4. The evaluation of sums like $M_1$ dates back to Ingham’s [11] important work on $M(x)$ in which he considered sums of the form
\[
\sum_{0 < \gamma < T} (T - \gamma)^k \zeta'(\rho)^{-1}
\]
for $k \in \mathbb{R}$. The sum $M_2$ is of the form
\[
(2.6) \quad \sum_{0 < \gamma < T} |A(\rho)|^2 \quad \text{where} \quad A(s) = \sum_{n \leq \xi} a_n n^{-s}
\]
is a Dirichlet polynomial with $\xi \leq T$. Such sums have played an important role in various applications. For instance, results concerning the distribution of consecutive zeros of $\zeta(s)$ and discrete mean values of the zeta-function and its derivatives are proven in [1, 2, 3, 6, 12, 16, 19]. In each of these articles, the evaluation of the discrete mean (2.6) either makes use of the Guinand-Weil explicit formula or of Gonek’s uniform version [6] of Landau’s formula
\[
(2.7) \quad \sum_{0 < \gamma < T} x^{\beta+i\gamma} = -\frac{T}{2\pi} \Lambda(x) + E(x, T)
\]
for $x, T > 1$ where $E(x, T)$ is an explicit error function uniform in $x$ and $T$. A novel aspect of our approach is that it does not require the use of the Guinand-Weil explicit formula or of the Landau-Gonek explicit formula (2.7). Instead we evaluate $M_2$ using the residue theorem and a version of Montgomery and Vaughan’s mean value theorem for Dirichlet polynomials [14]. Our approach is simpler and it is likely that it can be extended to evaluate the discrete mean (2.6) for a large class of coefficients $a_n$ with $\xi \leq T$.

3. The estimation of $M_1$

To estimate $M_1$, we require the following version of Montgomery and Vaughan’s mean value theorem for Dirichlet polynomials.
Lemma. Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of complex numbers. For any real number \( T > 0 \), we have

\[
\int_0^T \left( \sum_{n=1}^{\infty} a_n n^{-it} \right) \left( \sum_{n=1}^{\infty} b_n n^{it} \right) dt = T \sum_{n=1}^{\infty} a_n b_n + O \left( \left( \sum_{n=1}^{\infty} n |a_n|^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} n |b_n|^2 \right)^{\frac{1}{2}} \right).
\]

(3.1)

Proof. This is Lemma 1 of Tsang [22]. The special case where \( b_n = a_n \), is originally due to Montgomery and Vaughan [14]. It turns out, as shown by Tsang, that this special case is equivalent to the more general case stated in the lemma. \( \square \)

Let \( T \geq 4 \) and set \( c = 1 + (\log T)^{-1} \). It is well known (see Theorem 14.16 of Titchmarsh [21]) that assuming the Riemann Hypothesis there exists a sequence \( T = \{\tau_n\}_{n=3}^{\infty}, \) \( n < \tau_n \leq \tau_n + 1 \), and a fixed constant \( A > 0 \) such that

\[
\left| \zeta(\sigma+i\tau_n) \right|^{-1} \ll \exp \left( \frac{A \log \tau_n}{\log \log \tau_n} \right)
\]

uniformly for \( \frac{1}{2} \leq \sigma \leq 2 \). We now prove the estimate (2.4) assuming that \( T \in T \). Recall that \( |\gamma| > 1 \) for every non-trivial zero \( \rho = \frac{1}{2} + i \gamma \) of \( \zeta(s) \). Thus, assuming that all the zeros of \( \zeta(s) \) are simple, the residue theorem implies that

\[
M_1 = \frac{1}{2\pi i} \left( \int_{c+i}^{c+iT} + \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1-c+i} + \int_{1-c+i}^{c+i} \right) \frac{1}{\zeta(s)} M_{\xi}(1-s) \frac{1}{\xi} ds
\]

\[
= I_1 + I_2 + I_3 + I_4.
\]

say. Here we are using the fact that the residue of the function \( 1/\zeta(s) \) at \( s = \rho \) equals \( 1/\zeta'(\rho) \) if \( \rho \) is a simple zero of \( \zeta(s) \).

The main contribution to \( M_1 \) comes from the integral \( I_1 \); the remainder of the integrals contribute an error term. Observe that

\[
I_1 = \frac{1}{2\pi} \int_1^T \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{c+iT}} \sum_{n \leq \xi} \frac{\mu(n)}{n^{1-c-iT}} dt.
\]

By (3.1) with \( a_m = \mu(m)m^{-c} \) and \( b_n = \mu(n)n^{-1-c} \) it follows that

\[
I_1 = \frac{(T - 1)}{2\pi} \sum_{n \leq \xi} \frac{\mu(n)^2}{n} + O \left( \left( \sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^{2\xi-1}} \right)^{\frac{1}{2}} \left( \sum_{n \leq \xi} \mu(n)^2 n^{2\xi-1} \right)^{\frac{1}{2}} \right).
\]

Since

\[
\sum_{n \leq \xi} \frac{\mu(n)^2}{n} = \frac{6}{\pi^2} \log \xi + O(1),
\]

(3.3)
we conclude that
\[ I_1 = \frac{3}{\pi^3} T \log \xi + O\left(\xi \sqrt{\log T} + T\right) \]
for our choice of \(c\). Here we have used the fact that
\[ \sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^{2c-1}} \leq \zeta(2c - 1) \ll \log T. \]

To estimate the contribution from the integral \(I_2\), we recall the functional equation for the Riemann zeta-function which says that
\[ (3.4) \quad \zeta(s) = \chi(s)\zeta(1-s) \]
where
\[ \chi(s) = 2^s\pi^{s-1}\Gamma(1-s) \sin\left(\frac{\pi s}{2}\right). \]
Stirling’s asymptotic formula for the Gamma-function can be used to show that
\[ (3.5) \quad |\chi(\sigma+it)| = \left(\frac{|t|}{2\pi}\right)^{1/2-\sigma} \left(1 + O(|t|^{-1})\right) \]
uniformly for \(-1 \leq \sigma \leq 2\) and \(|t| \geq 1\). Combining this estimate and (3.2), it follows that, for \(T \in \mathcal{T}\),
\[ |\zeta(\sigma + iT)|^{-1} \ll T^{\min(\sigma-1/2,0)} \exp\left(\frac{A \log T}{\log \log T}\right) \]
uniformly for \(-1 \leq \sigma \leq 2\). In addition, we have the trivial bound
\[ (3.6) \quad |M_\xi(\sigma+it)| \ll \xi^{1-\sigma}. \]
Thus, estimating the integral \(I_2\) trivially, we find that
\[ I_2 \ll \exp\left(\frac{A \log T}{\log \log T}\right) \int_1^c T^{\min(\sigma-1/2,0)} \xi^\sigma d\sigma \ll \xi \exp\left(\frac{A \log T}{\log \log T}\right). \]

To bound the contribution from the integral \(I_3\), we notice that the functional equation for \(\zeta(s)\) combined with the estimate in (3.5) implies that, for \(1 \leq |t| \leq T\),
\[ |\zeta(1-c+it)|^{-1} \ll |t|^{1/2-c} |\zeta(c-it)|^{-1} \ll |t|^{1/2-c}\zeta(c) \ll |t|^{-1/2} \log T. \]
It therefore follows that
\[ I_3 \ll \log T \left(\sum_{n \leq \xi} \frac{\mu(n)}{n^c}\right) \int_1^T t^{-1/2} dt \ll \sqrt{T(\log T) \log \xi}. \]

Finally, since \(1/\zeta(s)\) and \(M_\xi(1-s)\) are bounded on the interval \([1-c+i, c+i]\), we find that \(I_4 \ll 1\). Hence, our combined estimates for \(I_1, I_2, I_3,\) and \(I_4\) imply that
\[ M_1 = \frac{3}{\pi^3} T \log \xi + O\left(\xi \exp\left(\frac{A \log T}{\log \log T}\right) + T\right). \]
From this and (2.1), the estimate in (2.4) follows.
4. THE ESTIMATION OF $M_2$

We now turn our attention to estimating the sum $M_2$. As before, let $T \geq 4$ and $c = 1 + (\log T)^{-1}$. Assuming the Riemann Hypothesis, we notice that

$$M_2 = \sum_{0 < \gamma \leq T} M_\xi(\rho)M_\xi(1-\rho).$$

Therefore, by the residue theorem, we see that

$$M_2 = \frac{1}{2\pi i} \left( \int_{c+i}^{c+iT} + \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1-c+i} + \int_{1-c+i}^{c+i} \right) M_\xi(s)M_\xi(1-s)\frac{\zeta'(s)}{\zeta(s)} ds$$

$$= J_1 + J_2 + J_3 + J_4,$$

say. In order to evaluate the integrals over the horizontal part of the contour we shall impose some extra conditions on $T$. Without loss of generality, we may assume that $T$ satisfies

$$|\gamma - T| \gg \frac{1}{\log T} \quad \text{for all ordinates } \gamma \text{ and}$$

$$\frac{\zeta'}{\zeta}(\sigma+iT) \ll (\log T)^2 \quad \text{uniformly for all } 1-c \leq \sigma \leq c.$$  \hspace{1cm} (4.1)

In each interval of length one such a $T$ exists. This well-known argument may be found in [4], page 108. Applying (3.6) we find

$$\sum_{T < \gamma < T+1} |M_\xi(\rho)M_\xi(1-\rho)| \ll \xi(\log T).$$

Therefore our choice of $T$ determines $M_2$ up to an error term $O(\xi \log T)$.

First we estimate the horizontal portions of the contour. By (3.6) and (4.1), we have

$$J_2 = \frac{1}{2\pi} \int_c^{1-c} M_\xi(\sigma+it)M_\xi(1-\sigma-it)\frac{\zeta'}{\zeta}(\sigma+it) d\sigma \ll \xi(\log T)^2.$$

Similarly, it may be shown that $J_4 \ll \xi$. Next we relate $J_3$ to $J_1$. We have

$$J_3 = \frac{1}{2\pi} \int_T^1 M_\xi(1-c+it)M_\xi(c-it)\frac{\zeta'}{\zeta}(1-c+it) dt$$

$$= -\frac{1}{2\pi} \int_1^T M_\xi(1-c-it)M_\xi(c+it)\frac{\zeta'}{\zeta}(1-c-it) dt$$

By differentiating (3.4), the functional equation, we find that

$$-\frac{\zeta'}{\zeta}(1-c-it) = -\frac{\chi'}{\chi}(1-c-it) + \frac{\zeta'}{\zeta}(c+it)$$
and hence that
\[
J_3 = -\frac{1}{2\pi} \int_1^T M_\xi(1-c-it)M_\xi(c+it) \frac{\chi'}{\chi}(1-c-it) \, dt \\
+ \frac{1}{2\pi} \int_1^T M_\xi(1-c-it)M_\xi(c+it) \frac{\zeta'}{\zeta}(c+it) \, dt.
\]

By (3.4) and Stirling's formula it can be shown that
\[-\frac{\chi'}{\chi}(1-c-it) = \log \left( \frac{|t|}{2\pi} \right) \left( 1 + O(|t|^{-1}) \right)\]
uniformly for \(1 \leq |t| \leq T\). By (3.6), the term \(O(|t|^{-1})\) contributes to \(J_3\) an amount which is \(O(\xi \log T)\) and, hence, it follows that
\[J_3 = K + J_1 + O(\xi \log T)\]
where
\[K = \int_1^T \log \left( \frac{t}{2\pi} \right) M_\xi(c+it)M_\xi(1-c-it) \, dt.\]

Collecting estimates, we deduce that
\[(4.2) \quad M_2 = K + 2\Re J_1 + O(\xi \log T) + O(\xi \log T^2).\]

To complete our estimation of \(M_2\), it remains to evaluate \(K\) and then \(J_1\).

Integrating by parts, it follows that
\[
K = \frac{1}{2\pi} \log \left( \frac{T}{2\pi} \right) \int_1^T M_\xi(c+it)M_\xi(1-c-it) \, dt \\
- \frac{1}{2\pi} \int_1^T \left( \int_1^t M_\xi(c+iu)M_\xi(1-c-iu) \, du \right) \frac{dt}{t}.
\]

By (3.1), we have
\[
\int_1^t M_\xi(c+iu)M_\xi(1-c-iu) \, du = (t-1) \sum_{n \leq \xi} \frac{\mu(n)^2}{n} + O(\xi \sqrt{\log T}) \\
= \frac{6}{\pi^2} t \log \xi + O(\xi \sqrt{\log T} + t)
\]
for \(t > 1\). Substituting this estimate into the above expression for \(K\), we see that
\[(4.3) \quad K = \frac{3}{\pi^3} T \log \left( \frac{T}{2\pi} \right) \log \xi + O(T \log T + O(T \log \xi)) \]
\[= \frac{3}{\pi^3} T \log \left( \frac{T}{2\pi} \right) \log \xi + O(T \log T).\]

We finish by evaluating the integral \(J_1\) which is similar to the evaluation of the integral \(I_1\) in the previous section. By another application of (3.1), we
find that

\[ J_1 = -\frac{1}{2\pi} \int_1^T \sum_{n=1}^\infty \frac{\alpha_n}{n^{c+it}} \sum_{n \leq \xi} \mu(n) n^{1-c-it} \, dt = -\frac{(T-1)}{2\pi} \sum_{n \leq x} \frac{\alpha_n \mu(n)}{n} \]

\[ + O\left( \left( \sum_{n=1}^\infty \frac{\alpha_n^2}{n^{2c-1}} \right)^\frac{1}{2} \left( \sum_{n \leq \xi} \frac{\mu(n)^2}{n^{1-2c}} \right)^\frac{1}{2} \right) \]

where the coefficients \( \alpha_n \) are defined by

\[ \alpha_n = \sum_{k\ell=n} \Lambda(k)\mu(\ell). \]

Observe that trivially \(|\alpha_n| \leq \sum_{|n| \leq \xi} \Lambda(u) \leq \log n\). It follows that the error term in the above expression for \( J_1 \) is \( \ll \zeta''(2c-1) \xi \ll \xi (\log T)^{\frac{3}{2}} \).

Finally, we note that

\[ \sum_{n \leq x} \frac{\alpha_n \mu(n)}{n} = \sum_{\ell \leq \xi} \frac{\mu(\ell)}{\ell} \sum_{k \leq \frac{x}{\ell}} \frac{\Lambda(k)\mu(k\ell)}{k} = \sum_{\ell \leq \xi} \frac{\mu(\ell)}{\ell} \sum_{p \leq \frac{\xi}{\ell}} \frac{\mu(p\ell)}{p} \log p \]

\[ = \sum_{\ell \leq \xi} \frac{\mu(\ell)}{\ell} \sum_{p \leq \frac{\xi}{\ell}} \frac{\mu(p\ell)}{p} \log p + O(\log \xi) \]

\[ = -\sum_{\ell \leq \xi} \frac{\mu(\ell)^2}{\ell} \sum_{p \leq \frac{\xi}{\ell}} \log p + O\left( \log \xi + \sum_{\ell \leq \xi} \sum_{p | \ell} \frac{1}{p} \sum_{\ell' \leq \frac{x}{\ell}} \log p \right) \]

since \( \mu(p\ell) = -\mu(\ell) \) if \((p, \ell) = 1\) and \( \mu(p\ell) = 0 = O(1) \) if \( p | \ell \). The sum in the error term is

\[ \sum_{\ell \leq \xi} \sum_{p \leq \frac{\xi}{\ell}} \frac{\log p}{p} = \sum_{p \leq x} \frac{(\log p)}{p^2} \sum_{\ell' \leq \frac{x}{p}} \frac{1}{\ell'} \ll \log \xi. \]

Hence, by the elementary result \( \sum_{p \leq \xi} \frac{\log p}{p} = \log \xi + O(1), (3.3), \) and partial summation, we deduce that

\[ \sum_{n \leq x} \frac{\alpha_n \mu(n)}{n} = -\sum_{\ell \leq \xi} \frac{\mu(\ell)^2 \log(\frac{\xi}{\ell})}{\ell} + O(\log \xi) = -\frac{3}{\pi^2} (\log \xi)^2 + O(\log \xi). \]

Therefore, combining formulae, we have

\[ (4.4) \quad J_1 = -\frac{3}{2\pi^3} T (\log \xi)^2 + O(T \log T). \]

Finally \((4.2), (4.3), \) and \((4.4) \) imply that

\[ M_2 = \frac{3}{\pi^3} T \log T \log \xi + \frac{3}{\pi^3} T (\log \xi)^2 + O(T \log T) \]

and, thus, by \((2.1)\) we deduce \((2.3)\).
References


