THE DISTRIBUTION OF VALUES OF THE SUMMATORY FUNCTION OF THE MÖBIUS FUNCTION

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July 2002

This note summarizes the part of my CMS doctoral prize lecture that focussed on the summatory function of the Möbius function. The lecture was titled "Limiting distributions and zeros of Artin L-functions" and was presented in Toronto at the CMS Winter meeting in December 2001.

I wish to thank the Canadian Mathematical Society for the honour of being chosen as the recipient of the 2001 CMS doctoral prize. I would like to thank my supervisor, Professor David Boyd, who diligently guided me, generously gave of his time, and shared of his extensive knowledge of mathematics.

The summatory function of the Möbius function.

The Möbius function is defined as the generating sequence for the reciprocal of the Riemann zeta function, that is,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \ . \tag{1}$$

This translates to $\mu(n) = (-1)^k$ if $n = p_1 \dots p_k$ is squarefree and $\mu(n) = 0$ otherwise. The Möbius function plays an important role in the analytic theory of numbers. It is especially important in sieve theory and in the method of mollification as initiated by Selberg in his study of the zeros of the Riemann zeta function on the critical line.

By partial summation of (1), we obtain

$$\frac{1}{\zeta(s)} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}}\, dx \tag{2}$$

valid for $Re(s) > 1$ where

$$M(x) = \sum_{n \le x} \mu(n) \tag{3}$$

The author was supported in part by a grant from NSERC and NSF grant DMS 9729992.

is the summatory function of the Möbius function. The identity (2) demonstrates the direct connection between the zeta function and $M(x)$. Over the years, this function has been much studied and speculated about. One reason for interest in $M(x)$ is that the Riemann hypothesis is equivalent to the bound

$$|M(x)| \ll x^{\frac{1}{2}} \exp\left(c\frac{\log x}{\log\log x}\right)$$

for an effective constant c and any $\epsilon > 0$. Moreover, Stieltjes and Mertens conjectured

$$|M(x)| \le x^{\frac{1}{2}} \tag{4}$$

for all $x > 1$. Mertens based this conjecture on a numerical calculation of $M(n)$ for $n = 1 \ldots 10000$. A related conjecture, known as the weak Mertens conjecture, asserts that

$$\int_1^X \left(\frac{M(x)}{x}\right)^2 dx \ll \log X\ . \tag{5}$$

Each of these conjectures imply the Riemann hypothesis and the simplicity of all of the zeros of $\zeta(s)$. For a time, it was believed that the bounds (4),(5) were true. However, Ingham [In] dispelled the notion that (4) could be true with a conditional proof that

$$\limsup_{x\to\infty} x^{-\frac{1}{2}}M(x) = \infty\ ,\ \liminf_{x\to\infty} x^{-\frac{1}{2}}M(x) = -\infty \tag{6}$$

assuming certain statistical properties of the zeros of $\zeta(s)$. Following Ingham's ideas, Odlyzko and te Riele [OR] proved unconditionally in 1986 that

$$\limsup_{x\to\infty} x^{-\frac{1}{2}}M(x) > 1.06\ ,\ \liminf_{x\to\infty} x^{-\frac{1}{2}}M(x) < -1.009\ .$$

The question we now address is what is the true behaviour of $M(x)$? Odlyzko and te Riele write in their article that "No good conjectures about the rate of growth of $M(x)$ are known." We first present the current state of knowledge regarding $M(x)$. The best unconditional upper bound is

$$|M(x)| \ll x\exp(-c\log^{\frac{3}{5}} x(\log\log x)^{-\frac{1}{5}})$$

for some effective constant c. On the other hand, if the Riemann hypothesis is false, then

$$M(x) = \Omega_{\pm}(x^{\theta-\epsilon})$$

where $\theta = \sup Re(\rho)$ with ρ ranging over the non-trivial zeros of $\zeta(s)$ and any $\epsilon > 0$. Also observe that the existence of a multiple zero would drastically change the expected behaviour of $M(x)$. For example if $\Theta + i\gamma$ were a multiple zero of order $m \geq 1$ then

$$M(x) = \Omega_{\pm}(x^{\Theta} \log^{m-1} x) \ .$$

Since we have some understanding of the behaviour of $M(x)$ in these unlikely scenarios we assume the opposite is true. Namely, we assume the Riemann hypothesis is true and that all zeros of the zeta function are simple. This is the most interesting case to consider and also the more difficult case. It is currently known [C] that at least $\frac{4}{10}$ of the zeros are simple and lie on the critical line.

Our main interest in this problem originated with a comment of Heath-Brown [HB]. He writes, "It appears to be an open question whether

$$x^{-\frac{1}{2}} M(x)$$

has a distribution function. To prove this one would want to assume the Riemann hypothesis and the simplicity of the zeros, and perhaps also a growth condition on $M(x)$." The key point is to construct a distribution function (probability measure) that demonstrates the properties of $M(x)$. Our approach to this problem is to exploit the connection between $M(x)$ and negative discrete moments of the Riemann zeta function.

Discrete moments of the Riemann zeta function.

Inverting equation (2) (by Perron's formula) we have

$$M(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s\zeta(s)} \, ds$$

where $c > 1$ and $x \notin \mathbb{Z}$. Moving the contour to the left, it follows that

$$M(x) = \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho \zeta'(\rho)} + E(x, T)$$

where $E(x, T)$ is a suitable error term. This last identity makes it clear that information concerning the sum

$$J_k(T) = \sum_{0 < \gamma < T} |\zeta'(\rho)|^{2k}$$

with $k < 0$ would be especially useful in obtaining information concerning $M(x)$. In the 1980's, these discrete moments began to attract interest. Gonek [Go1] considered Dirichlet polynomial approximations of these sums and Hejhal studied [Hej] the value distribution of $\log \zeta'(\rho)$. From these different points of view, Gonek and Hejhal independently conjectured that

$$J_k(T) \asymp T(\log T)^{(k+1)^2} \ . \tag{7}$$

Recently, [HKO] using random matrix model techniques have refined this conjecture. They conjecture that

$$J_k(T) \sim \frac{G^2(k+2)}{G(2k+3)} a_k \frac{T}{2\pi}\left(\log \frac{T}{2\pi}\right)^{(k+1)^2} \tag{8}$$

for $k > -\frac{3}{2}$ where

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \left(\sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}\right)$$

and G is Barnes' function defined by

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{1}{2}(z^2 + \gamma z^2 + z)\right) \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n}\right)^n e^{-z+z^2/2n}\right) .$$

This conjecture has been proven for $k = 0$ by Von Mangoldt and for $k = 1$ by Gonek assuming the Riemann hypothesis. In the case $k = 2$ the author has proven that the Riemann hypothesis implies this is the correct order of magnitude with explicit upper and lower bounds. As for the negative moments, less is known. Gonek established a conditional proof that $J_{-1}(T) \gg T$ and also conjectured that $J_{-1}(T) \sim \frac{3}{\pi^3} T$ which agrees with (8).

The idea of using $J_k(T)$ to study $M(x)$ was first realized by Gonek who makes use of these connections to study $M(x)$ in short intervals [Go2]. In order to obtain any information about $M(x)$, it is necessary to understand $J_k(T)$. Without any knowledge of $J_k(T)$, no new information concerning $M(x)$ can be gleaned.

The limiting distribution.

The main theorem in [Ng2] is the construction of the limiting distribution of the function

$$e^{-\frac{y}{2}} M(e^y) .$$

Precisely, we prove

Theorem 1. *The Riemann hypothesis and the conjectural bound $J_{-1}(T) \ll T$ imply that there exists a limiting distribution ν on $\mathbb{R}$ such that*

$$\lim_{Y\to\infty} \frac{1}{Y} \int_0^Y f(e^{-\frac{y}{2}} M(e^y))\, dy = \int_{\mathbb{R}} f(x)\, d\nu(x) \tag{9}$$

for all bounded continuous functions $f(x)$.

The idea of the proof is to use the construction from [RS]. In their work, they study the distribution functions associated to counting functions of primes in arithmetic progressions. The problem in extending [RS] is that there is little unconditional information concerning $\zeta'(\rho)^{-1}$. Thus it was necessary to assume bounds for the sum $J_{-1}(T)$ as suggested in equations (7),(8) . The key lemma was proven by using techniques from Cramér [Cr] in conjunction with $J_{-1}(T) \ll T$. (It should be noted that the trivial bound is $J_{-1}(T) \ll T^{3+\epsilon}$ and this is not sufficient to obtain any significant results regarding $M(x)$.)

In a similar vein, we can prove bounds for $M(x)$ or averages of $M(x)$.

Theorem 2. The Riemann hypothesis and $J_{-1}(T) \ll T$ imply the weak Mertens conjecture in the form

$$\int_1^X \left(\frac{M(x)}{x}\right)^2 dx \sim \left(\sum_{\gamma>0} \frac{2}{|\rho\zeta'(\rho)|^2}\right) \log X \tag{10}$$

and

$$M(x) \ll x^{\frac{1}{2}} (\log\log x)^{\frac{3}{2}} \tag{11}$$

except on a set of finite logarithmic measure. (The exponent $\frac{3}{2}$ in (11) may be reduced to $\frac{5}{4}$ under the additional assumption $J_{-\frac{1}{2}}(T) \ll T(\log T)^{\frac{1}{4}}$.)

Speculations on the lower order.

We now illustrate how Theorem 1 is useful for studying the distribution of values of $M(x)$. Suppose Theorem 1 were true for the indicator function of the set $[V, \infty)$. Equation (9) would then read

$$\lim_{Y\to\infty} \frac{1}{Y} meas\{y \in [0, Y] \mid M(e^y)e^{-\frac{y}{2}} \geq V\} = \nu([V, \infty)) \ . \tag{12}$$

This indicates that the distribution of values of $M(x)$ is related to the tail of the probability measure ν. It is noted in [RS] that identity (12) would be true if ν is absolutely continuous. In fact, we can show that ν is absolutely continuous if the following conjecture is true.

Linear independence conjecture. Assume the Riemann hypothesis. If the nontrivial zeros of $\zeta(s)$ are denoted as $\rho = \frac{1}{2} + i\gamma$, then the positive imaginary ordinates γ are linearly independent over $\mathbb{Q}$.

Now consider the random variable

$$X(\underline{\theta}) = \sum_{\gamma_k>0} \frac{2}{|(\frac{1}{2}+i\gamma_k)\zeta'(\frac{1}{2}+i\gamma_k)|} \sin(2\pi\theta_k)$$

where $\underline{\theta} = (\theta_1, \theta_2, \dots ,)$ is an element of the infinite torus $\mathbb{T}^\infty$. We note that if P is the canonical probability measure on $\mathbb{T}^\infty$ then the linear independence conjecture implies

$$\nu(B) = P(X^{-1}(B))$$

where B is any Borel set in $\mathbb{R}$. Consequently, ν may be studied via X. Moreover, by assuming the linear independence conjecture and the assumptions of Theorem 1, we may compute the Fourier transform of ν exactly. It equals

$$\widehat{\nu}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} d\nu(t) = \prod_{\gamma>0} j_0\left(\frac{2\xi}{|(\frac{1}{2}+i\gamma)\zeta'(\frac{1}{2}+i\gamma)|}\right)$$

where

$$j_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{2m}}{(m!)^2}$$

is a Bessel function of order zero. Furthermore, by pursuing ideas of Montgomery [M] concerning sums of independent random variables we can show that for large V

$$\exp(-c_1 V^{c_2} \exp(c_3 V^{\frac{4}{5}})) \ll \nu([V,\infty)) \ll \exp(-c_4 V^{c_5} \exp(c_6 V^{\frac{4}{5}})) \tag{13}$$

for effective constants $c_i > 0$ for $i = 1 \dots 6$. These arguments assume the Riemann hypothesis, the linear independence conjecture and bounds for both $J_{-\frac{1}{2}}(T)$ and $J_{-1}(T)$. An analysis of these bounds suggest the following conjecture:

Conjecture. There exists a number $B > 0$ such that

$$\overline{\underline{lim}}_{x\to\infty} \frac{M(x)}{\sqrt{x}(\log\log\log x)^{\frac{5}{4}}} = \pm B \ . \tag{14}$$

After the completion of this work, I learned from Gonek that he had an argument to suggest this lower bound 20 years ago. Apparently the heuristic argument in [Ng2] of the lower bound (14) is similar to Gonek's, however it was discovered independently.

We note that in the prime number case, where $M(x)$ is replaced by $\psi(x) - x$, Montgomery [M] conjectures that the corresponding B equals $\frac{1}{2\pi}$. In this case, the value of B is not so clear and remains an open problem. We remark that years earlier Good and Churchhouse [GC] had conjectured that

$$\overline{\underline{lim}}_{x\to\infty} \frac{M(x)}{\sqrt{x}(\log\log x)^{\frac{1}{2}}} = \pm\frac{\sqrt{12}}{\pi} \ .$$

This was based on modelling $\mu(n)$ as a random sequence supported on the squarefree integers and by "applying" the law of the iterated logarithm. Although Good and Churchhouse's argument seems promising it relies on the assumption that $\mu(n)$ behaves randomly. However, the Möbius function is not a random sequence as it is connected directly to $\zeta(s)^{-1}$. Thus $M(x)$ is determined by the zeros of $\zeta(s)$ or more precisely by the discrete moments $J_{-\frac{1}{2}}(T), J_{-1}(T)$. Consequently, it is these negative moments that should determine the behaviour of $M(x)$.

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