The fourth moment of $\zeta^{'}(\rho)$

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Abstract

Discrete moments of the Riemann zeta function were studied by Gonek and Hejhal in the 1980's. They independently formulated a conjecture concerning the size of these moments. In 1999, Hughes, Keating, and O'Connell, by employing a random matrix model, made this conjecture more precise. Subject to the Riemann hypothesis, we establish upper and lower bounds of the correct order of magnitude in the case of the fourth moment.

1 Introduction

This article concerns discrete moments of the derivative of the Riemann zeta function of the form

$$J_k(T) = \sum_{0<\gamma\leq T} |\zeta^{'}(\rho)|^{2k}$$

where $\rho = \beta+i\gamma$ ranges over non-trivial zeros of $\zeta(s)$ and $k \in \mathbb{R}$. In particular, we focus on the case $k = 2$. These moments are discrete analogues of the ordinary moments of the Riemann zeta function. In recent years there has been renewed interest in the moments of L-functions, in part due to Keating and Snaith's [11] work in random matrix theory. Estimates for the discrete moments have number theoretic applications (see [2],[12],[13]). To date, few asymptotic formulae have been established for these moments. However, Gonek [5] and Hejhal [8] independently conjectured

$$J_k(T) \asymp T \log^{(k+1)^2} T \tag{1}$$

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for $k \in \mathbb{R}$. Hughes, Keating, and O'Connell [9], applying random matrix models refined this to:

Random Matrix Model Conjecture For $k > -\frac{3}{2}$ and bounded,

$$J_k(T) \sim \frac{G^2(k+2)}{G(2k+3)} \cdot a_k \cdot N(T) \cdot \left(\log \frac{T}{2\pi}\right)^{k(k+2)} \tag{2}$$

as $T \to \infty$,where G is Barnes' function defined by

$$G(z+1) = (2\pi)^{z/2} \exp\left(-\frac{1}{2}(z^2 + \gamma z^2 + z)\right) \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n}\right)^n e^{-z+z^2/2n}\right) ,$$

γ is Euler's constant, $a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}$, and $N(t)$ denotes the number of zeros of $\zeta(s)$ in the box with vertices $0, 1, 1+it, it$.

The number $a_2 = \zeta(2)^{-1} = \frac{6}{\pi^2}$ appears frequently in this article. Conjecture (2) agrees with results of Von Mangoldt and Gonek [3] in the cases $k = 0, 1$. Furthermore, one verifies $J_{-1}(T) \sim \frac{3}{\pi^3} T$ is the case $k = -1$. Gonek first conjectured this formula by methods similar to Montgomery's study of the pair correlation conjecture. When $k = 2$, (2) reduces to $J_2(T) \sim \frac{1}{2880\pi^3} T \log^9 T$. We establish that the random matrix theory conjecture is of the correct order of magnitude in this case. Throughout, we use the notation $L = \log \frac{T}{2\pi}$. Our main result is

Theorem 1 *The Riemann hypothesis implies*

$$\frac{c_1}{\pi^3} T L^9 \left(1 + O\left(\frac{\log L}{L}\right)\right) \leq J_2(T) \leq \frac{c_2}{\pi^3} T L^9 \left(1 + O\left(\frac{\log L}{L}\right)\right) \tag{3}$$

where

$$c_1 = (\sqrt{a} - \sqrt{b})^2 = 0.0000687... \ , \ c_2 = (\sqrt{a} + \sqrt{b})^2 = 0.0051561... \tag{4}$$

with $a = \frac{61}{60480}, b = \frac{97}{60480}$. In contrast, $\frac{1}{2880} = 0.0003472....$

The same techniques as Theorem 1, permit one to replace $\zeta'(s)$ by higher derivatives. We remark that only Theorem 1 depends on RH. All other lemmas, corollaries, and theorems are independent of any hypothesis. We establish the following unconditional result which may be of use in future moment calculations.

Theorem 2 *Let $d(n)$ denote the number of divisors of n and $\delta = \lambda/\log\left(\frac{T}{2\pi}\right)$ where $\lambda \in \mathbb{R}$ and $|\lambda| \ll 1$. Then we have*

$$\sum_{0<\gamma<T} \sum_{m \le \frac{\gamma}{2\pi}} \frac{d(m)}{m^{\rho+i\delta}} \sum_{n \le \frac{\gamma}{2\pi}} \frac{d(n)}{n^{1-\rho-i\delta}} = \frac{3}{\pi^3} \left(\frac{1}{5!} - 4 \sum_{j \ge 1} \frac{(-1)^j \lambda^{2j}}{(5+2j)!} \right) T L^5 (1 + o(1)) \tag{5}$$

where $\rho = \beta + i\gamma$ ranges over non-trivial zeros of the zeta function with $0 < \gamma < T$. The $o(1)$ term is $(\log L)/L$.

Notation We work with Dirichlet series of the form

$$(-1)^{\mu+\nu} \zeta^{(\mu)}(s) \zeta^{(\nu)}(s) = \sum_{n=1}^{\infty} \frac{d^{(\mu,\nu)}(n)}{n^s} \tag{6}$$

where $\mu, \nu \in \mathbb{Z}_{\ge 0}$. Note that $d^{(\mu,\nu)}(n) = (\log^{\mu} * \log^{\nu})(n)$ where $*$ denotes convolution. Furthermore, we set $d^{(\mu)}(n) := d^{(\mu,0)}(n)$. The generalized divisor function $d_k(n)$ for $k > 0$ is defined by its generating function $\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}$. In this article the arithmetic functions

$$\begin{aligned} \alpha(n) &:= d^{(1,1)}(n) = (\log n)\, d^{(1)}(n) - d^{(2)}(n) \ , \\ \beta_t(n) &:= (a_t * a_t)(n) = l^2 d(n) - 2l \ d^{(1)}(n) + \alpha(n) \ , \end{aligned} \tag{7}$$

where $l = \log(\frac{t}{2\pi})$ and $a_t(n) = \log(\frac{t}{2\pi n})$ appear often. To simplify notation, we define for an arbitrary sequence $a(n,t)$ with $n \in \mathbb{Z}^+$ and $t \in \mathbb{R}$ the Dirichlet polynomial

$$D_a(\sigma + it) = \sum_{n \le \frac{t}{2\pi}} \frac{a(n,t)}{n^{\sigma+it}} \ . \tag{8}$$

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1.1 Proof of Theorem 1

We commence with the proof of Theorem 1 since the the argument is rather simple. This proof is subject to Corollary 1, a mean value result, which is a special case of Lemma 5. However, the proofs of Corollary 1 and Theorem 2 are deferred until later. We first state Corollary 1.

Corollary 1 *We have*

$$S_\alpha = \sum_{0<\gamma\leq T} D_\alpha(\rho)D_\alpha(1-\rho) = \frac{61}{60480\pi^3}TL^9 + O(TL^8\log L)\ , \tag{9}$$

$$S_\beta = \sum_{0<\gamma\leq T} D_{\beta_\gamma}(\rho)D_{\beta_\gamma}(1-\rho) = \frac{97}{60480\pi^3}TL^9 + O(TL^8\log L) \tag{10}$$

where $\rho = \beta + i\gamma$ ranges through the non-trivial zeros of the zeta function with $0 < \gamma < T$. Note that $D_\alpha(s)$ and $D_{\beta_\gamma}(s)$ are Dirichlet polynomials associated to $\alpha(n)$ and $\beta_\gamma(n)$ as defined by (8).

Proof of Theorem 1. The approximate functional equation we require is

$$\zeta^{'}(\sigma+it)^2 = \sum_{n\leq\frac{|t|}{2\pi}} \frac{\alpha(n)}{n^{\sigma+it}} + \chi^2(\sigma+it)\sum_{n\leq\frac{|t|}{2\pi}} \frac{\beta_t(n)}{n^{1-\sigma-it}} + O(\log^3 t) \tag{11}$$

where $\alpha(n)$ and $\beta_t(n)$ are defined by (7) and $\chi(s) = \pi^{s-\frac{1}{2}}\Gamma(\frac{1-s}{2})/\Gamma(\frac{s}{2})$ is the factor from the functional equation of the zeta function. It satisfies $\zeta(s) = \chi(s)\zeta(1-s)$ and $\chi(s)\chi(1-s) = 1$. Equation (11) is derived in [1] (Lemma 3 p.29). Let ρ denote a non-trivial zero of the Riemann zeta function. By (11) we have

$$\begin{aligned}\zeta^{'}(\rho)^2\zeta^{'}(1-\rho)^2 =& (D_\alpha(\rho) + \chi^2(\rho)D_{\beta_\gamma}(1-\rho) + O(l^3))\cdot \\ & (D_\alpha(1-\rho) + \chi^2(1-\rho)D_{\beta_\gamma}(\rho) + O(l^3))\end{aligned} \tag{12}$$

where $l = \log\gamma$. Summing (12) over zeros that satisfy $0 < \mathrm{Im}(\rho) < T$ yields

$$\sum_{0<\gamma<T} \zeta^{'}(\rho)^2\zeta^{'}(1-\rho)^2 = S_1 + 2\mathrm{Re}(S_2) + S_3 + S_4 \tag{13}$$

where $S_1 = S_\alpha + S_\beta$, $S_2 = \sum_{0<\gamma<T}\chi^2(1-\rho)D_\alpha(\rho)D_{\beta_\gamma}(\rho)$,

$$S_3 \ll L^3 \sum_{0<\gamma<T} \left(|D_\alpha(\rho)| + \left|\chi^2(1-\rho)D_{\beta_\gamma}(\rho)\right|\right)\ ,$$

and $S_4 \ll (\log^6 T)N(T) \ll TL^7$. We have by Corollary 1

$$S_1 = S_\alpha + S_\beta = \frac{a+b}{\pi^3}TL^9 + O(TL^8\log L)$$

where $a = \frac{61}{60480}$ and $b = \frac{97}{60480}$. Note that under the assumption of RH $|\chi(1-\rho)| = 1$ and $|D_a(\rho)|^2 = D_a(\rho)D_a(1-\rho)$ for a real sequence $a = a(n,t)$. Hence assuming RH, Cauchy-Schwarz implies

$$|S_2| \leq S_\alpha^{\frac{1}{2}} S_\beta^{\frac{1}{2}} = \frac{\sqrt{ab}}{\pi^3} TL^9 \left(1 + O(L^{-1}\log L)\right)$$

and also $S_3 \ll (N(T)L^6)^{\frac{1}{2}} S_1^{\frac{1}{2}} \ll TL^{\frac{7}{2}} S_1^{\frac{1}{2}}$. Lastly we note that RH implies

$$|\zeta'(\rho)|^4 = \zeta'(\rho)^2 \zeta'(1-\rho)^2 \ . \tag{14}$$

By (13), (14), and collecting our estimates of the S_i for $i = 1, \ldots, 4$ we have

$$\frac{c_1}{\pi^3} TL^9(1 + O(L^{-1}\log L)) \leq J_2(T) \leq \frac{c_2}{\pi^3} TL^9(1 + O(L^{-1}\log L))$$

for c_1, c_2 as in (4) and Theorem 1 is established.

In the above calculation RH was used to evaluate S_2 and S_3 and to guarantee the identity (14). It may be possible, by more sophisticated techniques, to bound S_2 and S_3 independent of RH and obtain unconditional bounds for the sum in (13). Moreover, we expect S_2 to contribute to the main term of $J_2(T)$. In contrast, the analogous sum in Ingham's [10] calculation does not contribute.

2 Lemmas

Our calculations require an old formula of Landau's. We apply Gonek's uniform version (proven in [4] pp.401-403).

Lemma 1 *Let $x, T > 1$ then*

$$\sum_{0<\gamma\leq T} x^\rho = -\frac{T}{2\pi}\Lambda(x) + O\left(x(\log(2xT))(\log\log 3x)\right) \\ O\left((\log x)\min\left(T, \frac{x}{\langle x\rangle}\right)\right) + O\left((\log(2T))\min\left(T, \frac{1}{\log x}\right)\right) \tag{15}$$

where $\langle x\rangle$ denotes the distance from x to the nearest prime power other than x itself.

To prove Lemmas 3 and 5 we require estimates for divisor sums. We only need upper bounds for shifted divisor sums as in (i) below. Moreover, we do not require the stronger asymptotic formulae that have been proven. In addition, a Brun-Titchmarsh result for divisor sums is applied.

Lemma 2 (i) *If $r \leq x$ is a positive integer and $\sigma_{-1}(r) = \sum_{d|r} d^{-1}$ then*

$$\sum_{r<n\leq x} d(n)d(n-r) \ll \sigma_{-1}(r)x\log^2 x \ . \tag{16}$$

(ii) *Let $\lambda \in \mathbb{R}$, $k \in \mathbb{N}$, $a \in \mathbb{Z}$, $(a,k) = 1$ and $k < x^{1-\alpha}$ for any $\alpha > 0$, then*

$$\sum_{\substack{n\leq x \\ n\equiv a \bmod k}} d_r^{\lambda}(n) \ll \frac{x}{k}\left(\frac{\phi(k)}{k}\log x\right)^{r^{\lambda}-1} \ . \tag{17}$$

Proof. Part (i) is Lemma B2 of [10] p.296 and part (ii) is a direct application of Theorem 2 of [14] p.169.

We prove a general mean value result for sequences which behave like $d(n)$. Extending the following result to $d_k(n)$ for $k \geq 3$ would require knowledge of sums like (16) with $d(n)$ replaced by $d_k(n)$. However, such results have not been proven yet.

Lemma 3 *Suppose two sequences $a(n)$and $b(n)$ satisfy $a(n) \ll \log^A(n)d(n)$ and $b(n) \ll \log^B(n)d(n)$ for $A, B > 0$. Then we define for $\delta \in \mathbb{R}$ the mean values*

$$\begin{aligned} I = I(a,b;T,\delta) &= \sum_{0<\gamma<T} D_a\left(\rho + i\delta\right) D_b\left(1-\rho-i\delta\right) \ , \\ I(a,b;T) &:= I(a,b;T,0) \end{aligned} \tag{18}$$

and we have

$$\begin{aligned} I(a,b;T,\delta) = \frac{T}{2\pi}\Bigg(&\log\left(\frac{T}{2\pi}\right)\sum_{n\leq\frac{T}{2\pi}}\frac{a(n)b(n)}{n} - \sum_{mj\leq\frac{T}{2\pi}}\frac{\Lambda(j)a(m)b(mj)}{mj^{1-i\delta}} \\ &- \sum_{mj\leq\frac{T}{2\pi}}\frac{\Lambda(j)b(m)a(mj)}{mj^{1+i\delta}}\Bigg) + O\left(TL^{A+B+4}\log L\right) \ . \end{aligned} \tag{19}$$

Proof. By swapping summation order

$$I = \sum_{m \le \frac{T}{2\pi}} \sum_{n \le \frac{T}{2\pi}} \frac{a(m)b(n)}{n} \left(\frac{n}{m}\right)^{i\delta} \sum_{2\pi \max(m,n) \le \gamma \le T} \left(\frac{n}{m}\right)^{\rho} . \tag{20}$$

We decompose $I = I_1 + I_2 + I_3$ where

$$I_1 = \sum_{m \le \frac{T}{2\pi}} \frac{a(m)b(m)}{m} (N(T) - N(2\pi m)) , \tag{21}$$

$$I_2 = \sum_{m \le \frac{T}{2\pi}} \sum_{m<n} \frac{a(m)b(n)}{n} \left(\frac{n}{m}\right)^{i\delta} \sum_{2\pi n \le \gamma \le T} \left(\frac{n}{m}\right)^{\rho} , \tag{22}$$

and I_3 is the remaining piece consisting of terms with $n < m$. The second expression in (21) is $\ll L^{A+B+1} \sum_{m \le T} d^2(m) \ll TL^{A+B+4}$ and since $N(T) = \frac{TL}{2\pi} + O(T)$ we deduce

$$I_1 = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) \sum_{m \le \frac{T}{2\pi}} \frac{a(m)b(m)}{m} + O(TL^{A+B+4}) . \tag{23}$$

Note that for $u \in \mathbb{R}$ and $0 < C < T$

$$\overline{\sum_{C<\gamma<T} u^{\rho}} = u \sum_{C<\gamma<T} \left(\frac{1}{u}\right)^{\rho} \tag{24}$$

which follows from the symmetry of the zeros about $\mathrm{Re}(s) = \frac{1}{2}$. Consequently, we deduce

$$\overline{I_3} = \sum_{m \le \frac{T}{2\pi}} \sum_{n<m} \frac{a(m)b(n)}{m} \left(\frac{m}{n}\right)^{i\delta} \sum_{2\pi m \le \gamma \le T} \left(\frac{m}{n}\right)^{\rho} . \tag{25}$$

This expression has the same form as I_2 except the roles of $a(n)$ and $b(n)$ have been switched. Thus the evaluation of I_3 follows along similar lines to I_2. Putting $x = \frac{n}{m}$ and noticing $n \ll T$, (15) implies

$$\begin{aligned} \sum_{2\pi n \le \gamma \le T} x^{\rho} = \left(-\frac{T}{2\pi} + n\right) \Lambda(x) + O\left(x(\log(2xT))(\log\log 3x)\right) \\ + O\left((\log x) \min\left(T, \frac{x}{\langle x \rangle}\right)\right) + O\left((\log(2T)) \min\left(T, \frac{1}{\log x}\right)\right) . \end{aligned} \tag{26}$$

By inserting (26) into the inner sum of (22) we obtain $I_2 = I_{21}+I_{22}+I_{23}+I_{24}$ where

$$I_{21} = \sum_{m\le \frac{T}{2\pi}} \sum_{\substack{m<n\le \frac{T}{2\pi}\\ m|n}} \frac{a(m)b(n)}{n} \left(\frac{n}{m}\right)^{i\delta} \left(-\frac{T}{2\pi}+n\right) \Lambda\left(\frac{n}{m}\right) \tag{27}$$

and I_{22}-I_{24} correspond to the other terms in (26). Applying $d(uv)\le d(u)d(v)$ the second part of this expression is

$$\ll L^{A+B} \sum_{jm\le \frac{T}{2\pi}} d(m)d(mj)\Lambda(j) \ll TL^{A+B+3} \sum_{j\le T} \frac{\Lambda(j)d(j)}{j} \ll TL^{A+B+4}$$

since the final sum is $\ll \sum_{p\le T} \frac{\log p}{p}$. Consequently, we deduce that

$$I_{21} = -\frac{T}{2\pi} \sum_{mj\le \frac{T}{2\pi}} \frac{\Lambda(j)a(m)b(mj)}{mj^{1-i\delta}} + O\left(TL^{A+B+4}\right) \ . \tag{28}$$

The next term is

$$\begin{aligned} I_{22} &\ll \sum_{n\le \frac{T}{2\pi}} \sum_{m<n} \frac{a(m)b(n)}{n} \left(\frac{n}{m}\log\log\left(\frac{n}{m}\right)\left(\log\left(2\frac{n}{m}T\right)\right)\right) \\ &\ll TL^{A+B+1}\log L \sum_{n\le \frac{T}{2\pi}} d(n) \sum_{m<n} \frac{d(m)}{m} \ll TL^{A+B+4}\log L. \end{aligned} \tag{29}$$

The third term, I_{23}, is bounded by

$$\sum_{n\le \frac{T}{2\pi}} \sum_{m<n} \frac{a(m)b(n)}{n} \log\left(\frac{n}{m}\right) \left(\min\left(T, \frac{\frac{n}{m}}{\langle \frac{n}{m}\rangle}\right)\right) \ll L^{A+B} \sum_{m<n\le \frac{T}{2\pi}} \frac{d(m)}{m} \frac{d(n)\log\frac{n}{m}}{\langle \frac{n}{m}\rangle} \ . \tag{30}$$

In the last sum in (30), pairs (m,n) such that $\langle \frac{n}{m}\rangle > \frac{1}{4}$ contribute

$$\ll L^{A+B} \left(\sum_{m<\frac{T}{2\pi}} \frac{d(m)}{m}\right) \left(\sum_{n\le \frac{T}{2\pi}} d(n)\log n\right) \ll TL^{A+B+4} \ .$$

The remaining pairs satisfy $\langle \frac{n}{m}\rangle \le \frac{1}{4}$. For each pair (m,n) with $m<n$ we uniquely write $n = qm+r$ with $-\frac{m}{2} < r \le \frac{m}{2}$ and $q = \lfloor \frac{n}{m}\rfloor$ or $q = \lfloor \frac{n}{m}\rfloor + 1$.

By the identity

$$\left\langle \frac{n}{m} \right\rangle = \left\langle q + \frac{r}{m} \right\rangle = \begin{cases} \frac{|r|}{m} & \text{if} \quad q = p^k \text{ and } r \neq 0 \\ \geq \frac{1}{2} & \text{if} \quad q \neq p^k \text{ or } r = 0 \end{cases}$$

we need only consider q a prime power. Thus the contribution from those terms with $\langle \frac{n}{m} \rangle \leq \frac{1}{4}$ in the final sum in (30) is

$$\sigma := \sum_{m<X} d(m) \sum_{q \leq \frac{X}{m}+1,\ q=p^k} \log(q) \sum_{1 \leq |r| \leq \frac{m}{4}} \frac{d(qm+r)}{|r|} \tag{31}$$

where $X = \frac{T}{2\pi}$ and hence $I_{23} \ll \sigma L^{A+B} + TL^{A+B+4}$. Furthermore, we write $\sigma = \sigma_1 + \sigma_2 + \sigma_3$ according to the cases: (i) $q = p$ prime, (ii) $q = p^k,\ k \geq 2,\ X^{1-\delta} < p^k \leq X+1$, and (iii) $q = p^k,\ k \geq 2,\ p^k \leq X^{1-\delta}$ where δ is a small positive constant. The contribution from (i) is

$$\begin{aligned} \sigma_1 &\ll \sum_{m<X} d(m) \sum_{q \leq \frac{X}{m}+1} \Lambda(q) \sum_{1 \leq |r| \leq \frac{m}{4}} \frac{d(qm+r)}{|r|} \\ &\ll \sum_{n \leq X} d(n) \sum_{|r| \leq \frac{X}{4}, |r|<n} \frac{1}{|r|} \sum_{qm=n-r} \Lambda(q) d(m) \end{aligned} \tag{32}$$

where we wrote $n = qm + r$ and noticed that $qm \leq 2X$. Since $d(m) \leq d(qm) = d(n-r)$ and $\sum_{qm=n-r} \Lambda(q) = \log(n-r) \leq \log(2X) \ll L$, our sum is bounded by

$$\ll L \sum_{|r| \leq \frac{X}{4}} \frac{1}{|r|} \sum_{|r|<n \leq X} d(n) d(n-r) \ll TL^3 \sum_{r \leq \frac{X}{4}} \frac{\sigma_{-1}(r)}{r} \ . \tag{33}$$

The right-most inequality follows by Lemma 2 and thus

$$\sigma_1 \ll TL^3 \sum_{r \leq \frac{X}{4}} \frac{1}{r} \sum_{g|r} \frac{1}{g} = TL^3 \sum_{g \leq \frac{X}{4}} \frac{1}{g^2} \sum_{s \leq \frac{X}{4}} \frac{1}{s} \ll TL^4 \ . \tag{34}$$

Observe that in σ_2, condition (ii) implies $m \ll \frac{X}{q} \leq X^{\delta}$ and since $d(qm+r) \ll X^{\delta}$ we have

$$\begin{aligned} \sigma_2 &\ll \sum_{m \leq X^{\delta}} d(m) \sum_{p^k \ll \frac{X}{m},\ k \geq 2} \log(p^k) \sum_{r \leq \frac{m}{4}} \frac{d(qm+r)}{r} \\ &\ll X^{\delta} \log X \sum_{m \leq X^{\delta}} d(m) \sum_{p^k \ll \frac{X}{m},\ k \geq 2} \log(p^k) \ll T^{\frac{1}{2}+\delta} \ . \end{aligned} \tag{35}$$

In the final piece we have

$$\sigma_3 \ll \sum_{p^k \le X^{1-\delta},\ k\ge 2} \log(p^k) \sum_{|r|\le \frac{X}{2}} \frac{1}{|r|} \sum_{\substack{2|r|\le m<X \\ m\ll \frac{X}{p^k}\ ,\ p^k m+r<X}} d(m)d(p^k m+r)\ . \tag{36}$$

By Cauchy-Schwarz, the inner sum in (36) is

$$\ll \left(\frac{X}{p^k}\right)^{\frac{1}{2}} \log^{\frac{3}{2}} X \left(\sum_{\substack{n\ll X \\ n\equiv r \bmod p^k}} d(n)^2 \right)^{\frac{1}{2}}\ . \tag{37}$$

We now establish

$$\sum_{\substack{n\ll X \\ n\equiv r \bmod p^k}} d(n)^2 \ll \min(\mathrm{ord}_p(r), k)^2\ \frac{X(\log X)^3}{p^k}\ . \tag{38}$$

If $(r,p)=1$ (38) is true by (17). On the other hand, suppose $(r,p)>1$ and $r=p^u s$ with $(s,p)=1$. If $u\ge k$ then we have

$$\sum_{\substack{n\ll X \\ n\equiv 0 \bmod p^k}} d(n)^2 \le d(p^k)^2 \sum_{j\ll \frac{X}{p^k}} d(j)^2 \ll \frac{k^2}{p^k}(X\log^3 X)\ .$$

In the case $1\le u<k$, an analogous calculation establishes the other bound in (38). Combining (36), (37), and (38) we have

$$\sigma_3 \ll (X\log^4 X) \sum_{p^k\le X^{1-\delta},\ k\ge 2} \frac{k\log(p^k)}{p^k} \ll TL^4\ .$$

Putting together our estimates for the σ_i, we have $\sigma \ll TL^4$ and hence $I_{23}\ll TL^{A+B+4}$. Finally, I_{24} is

$$\ll L^{1+A+B} \sum_{n\le \frac{T}{2\pi}} \sum_{m<n} \frac{d(m)d(n)}{n} \frac{1}{\log\frac{n}{m}} \ll L^{1+A+B} \sum_{r<\frac{T}{2\pi}} \frac{1}{r} \sum_{r<n\le \frac{T}{2\pi}} d(n-r)d(n)\ .$$

Notice that the last sum was already treated in (33) and (34), so we have $I_{24} \ll TL^{A+B+4}$. Thus we arrive at

$$I_2 = -\frac{T}{2\pi} \sum_{mj \leq \frac{T}{2\pi}} \frac{\Lambda(j)a(m)b(mj)}{mj^{1-i\delta}} + O\left(TL^{A+B+4} \log L\right) \ . \tag{39}$$

Starting from (25) an analogous calculation demonstrates that

$$I_3 = -\frac{T}{2\pi} \sum_{mj \leq \frac{T}{2\pi}} \frac{\Lambda(j)b(m)a(mj)}{mj^{1+i\delta}} + O\left(TL^{A+B+4} \log L\right) \ . \tag{40}$$

Combining (23), (39), and (40) finishes the proof of the lemma.

In the next lemma, we evaluate the second and third sums of (19).

Lemma 4 *Suppose we have two sequences $a(n) \ll \log^A(n)d(n)$ and $b(n) \ll \log^B(n)d(n)$ which satisfy for each $p \leq t$*

$$\sum_{n \leq t} \frac{a(n)b(pn)}{n} = \sum_{u+v=\beta} s_{uv} \log^u p \log^v t + O\left(\log^{\beta-1} t + \frac{(\log^C p)(\log^\beta t)}{p}\right) \tag{41}$$

where β, C are positive absolute constants, $u, v \geq 0$, $s_{uv} \in \mathbb{C}$, and the implied constant in the error term depends only on $a(n)$ and $b(n)$. We associate to an expansion of the form (41) the constant

$$\mathcal{A}(a,b) = \sum_{u+v=\beta} s_{uv} \frac{u!v!}{(u+v+1)!} \ . \tag{42}$$

Then we have

$$\begin{aligned} M(a,b;X,\delta) &:= \sum_{mk \leq X} \frac{\Lambda(k)a(m)b(mk)}{k^{1-i\delta}m} \\ &= \tilde{L}^{\beta+1} \sum_{k=0}^{\infty} \frac{(i\delta\tilde{L})^k}{k!} \sum_{u+v=\beta} s_{uv} \frac{(u+k)!v!}{(u+v+k+1)!} + O(\tilde{L}^{\max(\beta, A+B+4)}) \end{aligned} \tag{43}$$

where $\tilde{L} = \log X$ and $\delta \in \mathbb{R}$. Moreover if $\delta = 0$, this reduces to

$$M(a,b;X,0) = \sum_{mk \leq X} \frac{\Lambda(k)a(m)b(mk)}{km} = \tilde{L}^{\beta+1}\mathcal{A}(a,b) + O(\tilde{L}^{\max(\beta, A+B+4)}) \ . \tag{44}$$

Proof. In the sum $M(a,b;X,\delta)$ the prime powers p^α with $\alpha \ge 2$ contribute

$$\sum_{p^\alpha \le X, \alpha \ge 2} \frac{\Lambda(p^\alpha)}{p^\alpha} \sum_{m < \frac{X}{p^\alpha}} \frac{a(m)b(mp^\alpha)}{m} \ll \tilde{L}^{A+B+4} \sum_{p^\alpha \le X,\ \alpha \ge 2} \frac{\alpha \Lambda(p^\alpha)}{p^\alpha} \ll \tilde{L}^{A+B+4} \ .$$

We arrive at

$$M(a,b;X,\delta) = \sum_{p \le X} \frac{\Lambda(p)}{p^{1-i\delta}} \sum_{m < \frac{X}{p}} \frac{a(m)b(mp)}{m} + O(\tilde{L}^{A+B+4}) \ .$$

We replace the inner sum above by the expression on the right side of (41). The contribution to $M(a,b;X,\delta)$ coming from the error term in (41) is

$$\tilde{L}^{\beta-1} \sum_{p \le X} \frac{\log p}{p} + \tilde{L}^\beta \sum_{p \ge 2} \frac{\Lambda(p) \log^C p}{p^2} \ll \tilde{L}^\beta \ .$$

This demonstrates that

$$M(a,b;X,\delta) = \sum_{u+v=\beta} s_{uv} \sum_{p \le X} \frac{\Lambda(p)}{p^{1-i\delta}} \log^u p \ \log^v \left(\frac{X}{p}\right) + O(\tilde{L}^{\max \beta, A+B+4}) \ . \tag{45}$$

By Stieltjes integration,

$$\sum_{p \le X} \frac{\Lambda(p)}{p^{1-i\delta}} \log^u p \ \log^v \left(\frac{X}{p}\right) = \int_1^X \log^u t \ \log^v \left(\frac{X}{t}\right) \ \frac{d\theta(t)}{t^{1-i\delta}} \tag{46}$$

where $\theta(t) = \sum_{p \le t} \log p$. The prime number theorem is $\theta(t) = t + O(t \exp(-c\sqrt{\log t}))$ and thus the main part of (46) equals

$$\int_1^X \log^u t \ \log^v \left(\frac{X}{t}\right) \ \frac{dt}{t^{1-i\delta}} = \sum_{k=0}^\infty \frac{(i\delta)^k}{k!} \int_1^X \log^{u+k} t \ \log^v \left(\frac{X}{t}\right) \ \frac{dt}{t} \tag{47}$$

$$= \tilde{L}^{\beta+1} \sum_{k=0}^\infty \frac{(i\delta\tilde{L})^k}{k!} \int_0^1 x^{u+k}(1-x)^v \ dx = \tilde{L}^{\beta+1} \sum_{k=0}^\infty \frac{(i\delta\tilde{L})^k}{k!} \frac{(u+k)!v!}{(u+v+k+1)!}$$

where we made the variable change $x = (\log t)/\tilde{L}$. The contribution arising from the error term in the prime number theorem is easily seen to be $\tilde{L}^\beta$.

Combining (45), (46), and (47) establishes the lemma.

Putting together Lemmas 3 and 4 we have the following computation of the main term of $I(a,b;T)$ in (18) subject to various conditions on the sequences $a(n)$ and $b(n)$.

Lemma 5 *Suppose we have two sequences $a(n) \ll \log^A(n)d(n)$ and $b(n) \ll \log^B(n)d(n)$ such that*

$$\sum_{n\leq t} \frac{a(n)b(n)}{n} = c_{a,b}\log^{\beta} t + O(\log^{\beta-1} t)\ , \tag{48}$$

$$\sum_{n\leq t} \frac{a(n)b(pn)}{n} = \sum_{u+v=\beta} s_{uv}\log^u p\log^v t + O\left(\log^{\beta-1} t + \frac{(\log^C p)(\log^{\beta} t)}{p}\right)\ , \text{ and} \tag{49}$$

$$\sum_{n\leq t} \frac{b(n)a(pn)}{n} = \sum_{u+v=\beta} t_{uv}\log^u p\log^v t + O\left(\log^{\beta-1} t + \frac{(\log^C p)(\log^{\beta} t)}{p}\right)\ , \tag{50}$$

where $c_{a,b}, A, B, \beta, C$ are fixed positive constants. Moreover, suppose that (49) and (50) hold for $p \leq t$ and the constant in the error term is independent of p. Then we have

$$I(a,b;T) = \frac{TL^{\beta+1}}{2\pi}\left(c_{a,b} - \mathcal{A}(a,b) - \mathcal{A}(b,a)\right) + O(T(L^{\max(\beta,A+B+4)})) \tag{51}$$

where $\mathcal{A}(a,b)$ and $\mathcal{A}(b,a)$ are constants defined by (42).

More notation For arbitrary sequences $a(n)$ and $b(n)$ define the functions

$$T_{a,b}(t) = \sum_{n\leq t} \frac{a(n)b(n)}{n} \text{ and } T_{a,b;p}(t) = \sum_{n\leq t} \frac{a(n)b(pn)}{n}\ . \tag{52}$$

Furthermore, we use the simplified notation

$$T_{\mu,\nu}(t) := T_{d^{(\mu)},d^{(\nu)}}(t) \text{ and } T_{\mu,\nu;p}(t) := T_{d^{(\mu)},d^{(\nu)};p}(t) \tag{53}$$

for $\mu,\nu \in \mathbb{Z}_{\geq 0}$. Also define

$$T_{(n_1,n_2),(n_3,n_4)}(t) := T_{d^{(n_1,n_2)},d^{(n_3,n_4)}(n)} = \sum_{n\leq t} \frac{d^{(n_1,n_2)}(n)d^{(n_3,n_4)}(n)}{n} \tag{54}$$

for $n_1, n_2, n_3, n_4 \in \mathbb{Z}_{\geq 0}$. Recall that $d^{(\mu,\nu)}(n)$ is defined by (6) and $d^{(\mu)}(n) = d^{(\mu,0)}(n)$. Note that $T_{(\mu,0),(\nu,0)}(t) = T_{\mu,\nu}(t)$. By Lemma 5, we need to evaluate sums of the form (52) in order to compute the constants $\mathcal{A}(a,b)$ in (42). Once this is done we obtain the main term asymptotic for $I(a,b;T)$ in (18).

Our calculations require an effective version of Perron's formula.

Lemma 6 *Let $F(s) := \sum_{n\geq 1} a_n n^{-s}$ be a Dirichlet series with finite abscissa of absolute convergence σ_a. Suppose there exists a real number $\alpha \geq 0$ such that*

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma} \ll (\sigma - \sigma_a)^{-\alpha} \ (\sigma > \sigma_a) \tag{55}$$

and that B is a non-decreasing function such that $|a_n| \leq B(n)$ for $n \geq 1$. Then for $x \geq 2, T \geq 2, \sigma \leq \sigma_a, \kappa := \sigma_a - \sigma + (\log x)^{-1}$, we have

$$\sum_{n\leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} F(s+w) \frac{x^w}{w}\, dw + O\left(\frac{x^{\sigma_a - \sigma} (\log x)^{\alpha}}{T} + \frac{B(2x)}{x^{\sigma}} \left(1 + x \frac{\log T}{T}\right)\right) . \tag{56}$$

Proof. This is Corollary 2.1 p.133 of [15].

The evaluation of (54) follows closely Theorem 7 of [7] pp.296-297.

Lemma 7 *We have $T_{(n_1,n_2),(n_3,n_4)}(t) = P(\log t) + O_\epsilon(t^{-\frac{1}{2}+\epsilon})$ where $P(x)$ is a polynomial of degree $n_1 + n_2 + n_3 + n_4 + 4$ with leading coefficient*

$$\frac{a_2 n_1! n_2! n_3! n_4!}{(n_1 + n_2 + n_3 + n_4 + 4)!} \sum_{a=0}^{n_1} \sum_{c=0}^{n_2} \binom{n_3 + 1 + a + c}{n_3} \binom{n_4 + 1 + n_1 + n_2 - a - c}{n_4} . \tag{57}$$

A special case of this result is $T_{\mu,\nu}(t) = Q(\log t) + O_\epsilon(t^{-\frac{1}{2}+\epsilon})$ where $Q(x)$ is a polynomial of degree $\mu + \nu + 4$ with leading coefficient

$$C(\mu,\nu) = \frac{\mu!\nu!}{(\mu+\nu+4)!} \left(\binom{\mu+\nu+2}{\mu+1} - 1 \right) . \tag{58}$$

Proof. Define $\sigma_{u,v}(n) = \sum_{d_1 d_2 = n} d_1^u d_2^v$ where $u, v \in \mathbb{C}$. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ and define the Dirichlet series

$$F(s; \vec{z}) := \sum_{n\leq t} \frac{\sigma_{-z_1,-z_2}(n) \sigma_{-z_3,-z_4}(n)}{n^{s+1}}$$

where $\vec{z} = (z_1, z_2, z_3, z_4)$. Observe the relationship

$$(-1)^{n_1+n_2+n_3+n_4} \frac{d^{n_1}}{dz_1^{n_1}} \frac{d^{n_2}}{dz_2^{n_2}} \frac{d^{n_3}}{dz_3^{n_3}} \frac{d^{n_4}}{dz_4^{n_4}} F(s;\vec{z})\Big|_{\vec{z}=\vec{0}} = \sum_{n\geq 1} \frac{d^{(n_1,n_2)}(n) d^{(n_3,n_4)}(n)}{n^{s+1}} \ . \tag{59}$$

We denote the generating function in (59) $F(s)$. On the other hand, by Ramanujan's calculation (see [16] pp.8-9), $F(s;\vec{z})$ equals

$$\frac{\zeta(1+s+z_2+z_4)\zeta(1+s+z_1+z_4)\zeta(1+s+z_2+z_3)\zeta(1+s+z_1+z_3)}{\zeta(2+2s+z_1+z_2+z_3+z_4)} \ . \tag{60}$$

By (59) and (60) we deduce that

$$F(s) := (-1)^N \sum_{\vec{a}\in(\mathbb{Z}_{\geq 0})^4} G_{\vec{a}}(s) \zeta^{(a_1)}(1+s)\zeta^{(a_2)}(1+s)\zeta^{(a_3)}(1+s)\zeta^{(a_4)}(1+s) \tag{61}$$

where $N = n_1+n_2+n_3+n_4$ and $\vec{a} = (a_1, a_2, a_3, a_4) \in (\mathbb{Z}_{\geq 0})^4$ ranges over a finite sum. Moreover, the functions $G_{\vec{a}}(s)$ have absolutely convergent Dirichlet series in $\mathrm{Re}(s) > -\frac{1}{2}$. A careful examination of (60) reveals that the leading term in the Laurent expansion of $F(s)$ derives from the expression

$$(-1)^N \frac{d^{n_1}}{dz_1^{n_1}} \frac{d^{n_2}}{dz_2^{n_2}} \frac{d^{n_3}}{dz_3^{n_3}} \frac{d^{n_4}}{dz_4^{n_4}} G(1+s;\vec{z}) \tag{62}$$

where

$$G(w;\vec{z}) := \zeta(w+z_2+z_4)\zeta(w+z_1+z_4)\zeta(w+z_2+z_3)\zeta(w+z_1+z_3) \ .$$

An application of the product rule $(f(z)g(z))^{(n)} := \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(z) g^{(n-j)}(z)$ in each of the variables to (62) yields

$$\begin{aligned} &(-1)^N \sum_{a,c,e,g} \binom{n_1}{a}\binom{n_2}{c}\binom{n_3}{e}\binom{n_4}{g} \zeta^{(a+e)}(1+s+z_1+z_3)\cdot \\ &\cdot \zeta^{(c+f)}(1+s+z_2+z_3)\zeta^{(b+g)}(1+s+z_1+z_4)\zeta^{(d+h)}(1+s+z_2+z_4) \end{aligned} \tag{63}$$

where $a+b = n_1, c+d = n_2, e+f = n_3$, and $g+h = n_4$. Thus

$$\begin{aligned} F(s) :=& \frac{(-1)^N}{\zeta(2+2s)} \sum_{a,c,e,g} \binom{n_1}{a}\binom{n_2}{c}\binom{n_3}{e}\binom{n_4}{g}\cdot \\ &\zeta^{(a+e)}(1+s)\zeta^{(c+f)}(1+s)\zeta^{(b+g)}(1+s)\zeta^{(d+h)}(1+s) + R(s) \end{aligned} \tag{64}$$

where $R(s)$ is a function with a pole of order at most $N+3$ at $s=0$. Note that we have the expansions $\zeta^{(k)}(1+s) = \frac{(-1)^k k!}{s^{k+1}} + c_k + \cdots$ and $\frac{1}{\zeta(2+2s)} = \frac{1}{\zeta(2)} + c^{'}s + \cdots$ for constants c_k and $c^{'}$. By combining (64) with these expansions, we have $F(s) = \frac{6}{\pi^2} C s^{-N-4} + R^{'}(s)$. Here $R^{'}(s)$ consists of those terms in the Laurent expansion with pole of order at most $N+3$ and

$$C = \sum_{a,c,e,g} \binom{n_1}{a}\binom{n_2}{c}\binom{n_3}{e}\binom{n_4}{g} \cdot (a+e)!(c+f)!(b+g)!(d+h)! \tag{65}$$

We simplify C by applying the identity

$$\sum_{k=0}^{l} \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1} \tag{66}$$

valid for integers $l, m \geq 0$ and integers $n \geq q \geq 0$ (see [6] p.169). The sum over g in (65) is

$$\begin{aligned} &\sum_{g=0}^{n_4} \binom{n_4}{g}(b+g)!(d+h)! = \sum_{g=0}^{n_4} \frac{n_4!}{g!(n_4-g)!}(b+g)!(d+n_4-g)! \\ &= n_4!b!d! \sum_{g=0}^{d+n_4} \binom{b+g}{b}\binom{d+n_4-g}{d} = n_4!b!d!\binom{n_4+b+d+1}{b+d+1} \end{aligned} \tag{67}$$

where we applied (66). Similarly, the sum over e is

$$\sum_{e=0}^{n_3} \binom{n_3}{e}(a+e)!(c+f)! = n_3!a!c!\binom{n_3+a+c+1}{a+c+1}. \tag{68}$$

Since $\binom{n_1}{a}a!b! = n_1!$ and $\binom{n_2}{c}c!d! = n_2!$ the total sum is

$$C = n_1!n_2!n_3!n_4! \sum_{a=0}^{n_1}\sum_{c=0}^{n_2} \binom{n_3+a+c+1}{n_3}\binom{n_4+1+n_1+n_2-a-c}{n_4}. \tag{69}$$

This shows that $F(s)s^{-1} := \frac{C}{s^{N+5}} + \frac{C'}{s^{N+4}} + \cdots$ for constants C and C'. Hence the residue of $F(s)t^s s^{-1}$ at $s=0$ is $P(\log t)$ where $P(t)$ is a polynomial of degree $N+4$ with leading coefficient $\frac{6}{\pi^2}C/(N+4)!$. By Lemma 6 applied with $\alpha = N+4$, $s = \sigma_a = 1$, and $B(t) \ll_\epsilon t^\epsilon$ it follows that

$$T(t) := \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} F(w)\frac{t^w}{w}\,dw + O\left(\frac{(\log t)^{N+4}}{T} + \frac{1}{t^{1-\epsilon}}\left(1 + t\frac{\log T}{T}\right)\right) \tag{70}$$

where $\kappa = (\log t)^{-1}$. By the residue theorem, the integral is

$$P(\log t) - \frac{1}{2\pi i}\left(\int_{\kappa+iT}^{c+iT} + \int_{c+iT}^{c-iT} + \int_{c-iT}^{\kappa-iT}\right)\frac{F(w)t^w}{w}\,dw\ . \tag{71}$$

where $c = -\frac{1}{2} + \epsilon$. We only sketch how to estimate these integrals since the argument is standard. The first and third integral may be computed by using known bounds for $\zeta(s)$ in the critical strip. The second integral requires the result

$$\int_1^U |\zeta^{(a)}(\tau + it)|^4 \ll_{a,\tau} U \tag{72}$$

for $a \in \mathbb{Z}_{\geq 0}$ and $\tau > \frac{1}{2}$. This may be proven by following the argument of Theorem 7.5 pp.146-147 of [16]. An appropriate choice of T then yields an error term of $t^{-\frac{1}{2}+\epsilon}$ to complete the proof. For the special case $T_{\mu,\nu}(t)$, we set $n_1 = \mu, n_2 = 0, n_3 = \nu$, and $n_4 = 0$. Applying the binomial identity (see [6] p.174)

$$\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n} \tag{73}$$

for $r, n \in \mathbb{Z}_{\geq 0}$, (57) reduces to

$$\frac{6}{\pi^2}\frac{\mu!\nu!}{(\mu+\nu+4)!}\sum_{a=0}^{\mu}\binom{\nu+1+a}{\nu} = \frac{6}{\pi^2}\frac{\mu!\nu!}{(\mu+\nu+4)!}\left(\binom{\mu+\nu+2}{\mu+1} - 1\right) \tag{74}$$

and thus (58) is verified.

We now record the special cases of Lemma 7 which are required in the proof of Corollary 1. In Table 1, we associate to each pair of sequences (a, b) the main term of $T_{a,b}(t)$ in (52).

Table 1

(a,b)	$T_{a,b}(t)$	(a,b)	$T_{a,b}(t)$
(d,d)	$\sim a_2 \cdot \frac{1}{24}l^4$	$(d^{(1)}, d^{(2)})$	$\sim a_2 \cdot \frac{1}{280}l^7$
$(d,d^{(1)})$	$\sim a_2 \cdot \frac{1}{60}l^5$	$(d^{(1)}, \alpha)$	$\sim a_2 \cdot \frac{1}{420}l^7$
$(d^{(1)}, d^{(1)})$	$\sim a_2 \cdot \frac{1}{144}l^6$	$(d^{(2)}, d^{(2)})$	$\sim a_2 \cdot \frac{19}{10080}l^8$
$(d,d^{(2)})$	$\sim a_2 \cdot \frac{1}{120}l^6$	(α,α)	$\sim a_2 \cdot \frac{17}{20160}l^8$
(d,α)	$\sim a_2 \cdot \frac{1}{180}l^6$		

We now evaluate $T_{d^{(\mu)},d^{(\nu)};p}(t)$ in (53).

Lemma 8 *Let $\mu, \nu \geq 0$ be integers and let p be any prime $\leq t$. We have*

$$T_{\mu,\nu;p}(t) = a_2 \cdot \left(2C(\mu,\nu) l^{\mu+\nu+4} + \sum_{k=0}^{\nu-1} \binom{\nu}{k} (\log^{\nu-k} p) l^{\mu+k+4} C(\mu,k) \right) + O_{\mu,\nu}\left(\frac{l^{\mu+\nu+4}}{p} + l^{\mu+\nu+3} \right) \tag{75}$$

where $l = \log t$, $C(\mu,k)$ is defined by (58), and the sum only occurs if $\nu \geq 1$ and is zero otherwise.

Proof. First note that

$$T_{\mu,\nu;p}(t) = \frac{d^\nu}{dz^\nu} \sum_{n \leq t} \frac{d^{(\mu)}(n)\sigma_z(pn)}{n} \Bigg|_{z=0} . \tag{76}$$

Inserting the identity

$$\sigma_z(n_1 n_2) = \sum_{m | (n_1,n_2)} \mu(m) m^z \sigma_z\left(\frac{n_1}{m}\right) \sigma_z\left(\frac{n_2}{m}\right)$$

in (76) and inverting summations we obtain

$$\sum_{n \leq t} \frac{d^{(\mu)}(n)\sigma_z(pn)}{n} = \sum_{j \leq t} \frac{d^{(\mu)}(j)\sigma_z(p)\sigma_z(j)}{j} - \frac{1}{p} \sum_{j \leq \frac{t}{p}} \frac{d^{(\mu)}(pj) p^z \sigma_z(j)}{j} . \tag{77}$$

Observe that

$$\frac{d^\nu}{dz^\nu} \sigma_z(p)\sigma_z(j) \Big|_{z=0} = \sum_{k=0}^{\nu-1} \binom{\nu}{k} \log^{\nu-k}(p) d^{(k)}(j) + 2d^{(\nu)}(j) \text{ and} \tag{78}$$

$$\frac{d^\nu}{dz^\nu} p^z \sigma_z(j) \Big|_{z=0} = \sum_{k=0}^{\nu} \binom{\nu}{k} \log^{\nu-k}(p) d^{(k)}(j) \tag{79}$$

where (79) follows from (78) since $\sigma_z(p) = 1 + p^z$. Combining (76), (77), (78), and (79) we deduce that $T_{\mu,\nu;p}(t)$ equals

$$2T_{\mu,\nu}(t) + \sum_{k=0}^{\nu-1} \binom{\nu}{k} (\log^{\nu-k} p)\ T_{\mu,k}(t) - \frac{1}{p} \left(\sum_{k=0}^{\nu} \binom{\nu}{k} (\log^{\nu-k} p)\, T_{k,\mu;p}\left(\frac{t}{p}\right) \right) . \tag{80}$$

The trivial bound $T_{k,\mu;p}(t) \ll (\log^{k+4} t)(\log^{\mu}(pt))$ follows from $d^{(k)}(j) \leq (\log^k j)d(j)$ and hence the error term is $\ll p^{-1}\log^{\nu+\mu+4} t$. Applying Lemma 7 to each expression in the main term of (80) completes the proof of the lemma.

We now compute $\mathcal{A}(d^{(\mu)}, d^{(\nu)})$.

Lemma 9 *Let $\mu, \nu \geq 0$ be integers. We have*

$$\mathcal{A}(d^{(\mu)}, d^{(0)}) = a_2 \cdot \frac{2}{(\mu+5)(\mu+4)(\mu+3)(\mu+2)} \tag{81}$$

and if $\nu \geq 1$

$$\mathcal{A}(d^{(\mu)}, d^{(\nu)}) = a_2 \cdot \frac{\mu!\nu!}{(\mu+\nu+5)!}\left(2\binom{\mu+\nu+2}{\nu+1} + \binom{\mu+\nu+2}{\nu} - \nu - 3\right). \tag{82}$$

Proof. By Lemma 8

$$\begin{aligned} T_{\mu,\nu;p}(t) = a_2 \cdot &\left(2C(\mu,\nu)l^{\mu+\nu+4} + \sum_{k=0}^{\nu-1}\binom{\nu}{k}(\log^{\nu-k} p)l^{\mu+k+4}C(\mu,k)\right) \\ &+ O\left(\frac{l^{\mu+\nu+4}}{p} + l^{\mu+\nu+3}\right) \end{aligned} \tag{83}$$

where $l = \log t$ and $C(\mu,k)$ is defined by (58). Hence by the definition (42) $\mathcal{A}(d^{(\mu)}, d^{(0)})$ equals

$$a_2 \cdot 2\frac{\mu!}{(\mu+4)!}\left(\binom{\mu+2}{\mu+1} - 1\right)\frac{(\mu+4)!}{(\mu+5)!} = a_2 \cdot \frac{2}{(\mu+5)(\mu+4)(\mu+3)(\mu+2)}$$

and if $\nu \geq 1$, $\mathcal{A}(d^{(\mu)}, d^{(\nu)})$ equals

$$a_2 \cdot \left(2\frac{C(\mu,\nu)}{\mu+\nu+5} + \sum_{k=0}^{\nu-1}\left(\frac{\nu!}{k!(\nu-k)!}\right)\left(\frac{(\nu-k)!(\mu+k+4)!}{(\mu+\nu+5)!}\right)C(\mu,k)\right). \tag{84}$$

By (73) the sum in (84) is

$$\frac{\mu!\nu!}{(\mu+\nu+5)!}\sum_{k=0}^{\nu-1}\left(\binom{\mu+k+2}{\mu+1} - 1\right) = \frac{\mu!\nu!}{(\mu+\nu+5)!}\left(\binom{\mu+\nu+2}{\nu} - \nu - 1\right). \tag{85}$$

Therefore (84) and (85) imply (82).

We summarize with a table of values of $\mathcal{A}(d^{(\mu)}, d^{(\nu)})$. In Table 2 the first column is the pair of sequences $(d^{(\mu)}, d^{(\nu)})$, the second column is the main term of $T_{\mu,\nu;p}(t)$ as in (75), and the third column is $\mathcal{A}(d^{(\mu)}, d^{(\nu)})$ as computed by (82). Here we use the notation $l = \log t$ and $u = \log p$.

Table 2

$(d^{(\mu)}, d^{(\nu)})$	main term of $T_{\mu,\nu;p}(t)$	$\mathcal{A}(d^{(\mu)}, d^{(\nu)})$
(d, d)	$a_2 \cdot \frac{1}{12} l^4$	$a_2 \cdot \frac{1}{60}$
$(d^{(1)}, d)$	$a_2 \cdot \frac{1}{30} l^5$	$a_2 \cdot \frac{1}{180}$
$(d^{(2)}, d)$	$a_2 \cdot \frac{1}{60} l^6$	$a_2 \cdot \frac{1}{420}$
$(d, d^{(1)})$	$a_2 \cdot (\frac{1}{30} l^5 + \frac{1}{24} l^4 u)$	$a_2 \cdot \frac{1}{144}$
$(d^{(1)}, d^{(1)})$	$a_2 \cdot (\frac{1}{72} l^6 + \frac{1}{60} l^5 u)$	$a_2 \cdot \frac{1}{420}$
$(d^{(2)}, d^{(1)})$	$a_2 \cdot (\frac{1}{140} l^7 + \frac{1}{120} l^6 u)$	$a_2 \cdot \frac{1}{960}$
$(d, d^{(2)})$	$a_2 \cdot (\frac{1}{60} l^6 + \frac{1}{30} l^5 u + \frac{1}{24} l^4 u^2)$	$a_2 \cdot \frac{1}{280}$
$(d^{(1)}, d^{(2)})$	$a_2 \cdot (\frac{1}{140} l^7 + \frac{1}{72} l^6 u + \frac{1}{60} l^5 u^2)$	$a_2 \cdot \frac{5}{4032}$
$(d^{(2)}, d^{(2)})$	$a_2 \cdot (\frac{19}{5040} l^8 + \frac{1}{140} l^7 u + \frac{1}{120} l^6 u^2)$	$a_2 \cdot \frac{5}{9072}$

Using the previous table we can compute $\mathcal{A}(a, b)$ for the remainder of the sequences we require for our calculation.

Lemma 10 *We have*

$$\mathcal{A}(\alpha, d) = a_2 \cdot \frac{1}{630} \ , \ \mathcal{A}(d, \alpha) = a_2 \cdot \frac{1}{420} \ , \ \mathcal{A}(\alpha, d^{(1)}) = a_2 \cdot \frac{1}{1440} \ , \tag{86}$$

$$\mathcal{A}(d^{(1)}, \alpha) = a_2 \cdot \frac{17}{20160} \ , \ \textit{and} \ \mathcal{A}(\alpha, \alpha) = a_2 \cdot \frac{23}{90720} \ . \tag{87}$$

Proof. We employ the following notation: if $A(t) = \sum_{n \le t} a_n$ and $j \in \mathbb{Z}_{\ge 0}$ then we define the operator $\mathcal{L}^j$ by $(\mathcal{L}^j A)(t) = \sum_{n \le t} (\log^j n)\ a_n$. Note that if $A(t) = \sum_{n \le t} a_n = \alpha \log^N t + O(\log^{N-1} t)$ then partial summation implies

$$(\mathcal{L}^j A)(t) = \alpha \frac{N}{N+j} \log^{N+j} t + O(\log^{N+j-1} t) \ . \tag{88}$$

By (7) we have the identities

$$\alpha(n) = \log n \ d^{(1)}(n) - d^{(2)}(n) \ , \ \alpha(pn) = (\log p + \log n) d^{(1)}(pn) - d^{(2)}(pn) \ . \tag{89}$$

We begin with one example. It follows from (89) that

$$d(n)\alpha(pn) = (\log p)\, d(n)d^{(1)}(pn) + (\log n)\, d(n)d^{(1)}(pn) - d(n)d^{(2)}(pn)$$

and hence

$$T_{d,\alpha;p}(t) = (\log p)\, T_{0,1;p}(t) + (\mathcal{L}T_{0,1;p})(t) - T_{0,2;p}(t)\ . \tag{90}$$

By Table 2, (88), and (90) we derive

$$T_{d,\alpha;p}(t) = uT_{0,1;p}(t) + (\mathcal{L}T_{0,1;p})(t) - T_{0,2;p}(t) = a_2 \cdot \left(\frac{l^6}{90} + \frac{l^5 u}{30}\right) + O\left(l^5 + \frac{l^6}{p}\right) \tag{91}$$

where $l = \log t$ and $u = \log p$. In a similar fashion we compute

$$\begin{aligned}
&T_{\alpha,d;p}(t) = (\mathcal{L}T_{1,0;p})(t) - T_{2,0;p}(t)\ ,\\
&T_{\alpha,d^{(1)};p}(t) = (\mathcal{L}T_{1,1;p})(t) - T_{2,1;p}(t)\ ,\\
&T_{d^{(1)},\alpha;p}(t) = (\log p)\, T_{1,1;p}(t) + (\mathcal{L}T_{1,1;p})(t) - T_{1,2;p}(t)\ ,\ \textit{and}\\
&T_{\alpha,\alpha;p}(t) = (\log p)\, (\mathcal{L}T_{1,1;p})(t) + (\mathcal{L}^2 T_{1,1;p})(t) - (\mathcal{L}T_{1,2;p})(t)\\
&-(\log p)\, T_{2,1;p}(t) - (\mathcal{L}T_{2,1;p})(t) + T_{2,2;p}(t)\ .
\end{aligned} \tag{92}$$

Thus Table 2, (88), and (92) imply

$$T_{a,b;p}(t) = \sum_{i+j=A} c_{ij} (\log^i t)(\log^j p) + O\left(\frac{\log^A t}{p} + \log^{A-1} t\right) \tag{93}$$

for the aforementioned sequences (a,b) and appropriate constants c_{ij}, A. In summary, we obtain

Table 3

(a,b)	main term of $T_{a,b;p}(t)$	$\mathcal{A}(a,b)$
(α,d)	$a_2 \cdot \frac{1}{90} l^6$	$a_2 \cdot \frac{1}{630}$
(d,α)	$a_2 \cdot (\frac{1}{90} l^6 + \frac{1}{30} l^5 u)$	$a_2 \cdot \frac{1}{420}$
$(\alpha,d^{(1)})$	$a_2 \cdot (\frac{1}{210} l^7 + \frac{1}{180} l^6 u)$	$a_2 \cdot \frac{1}{1440}$
$(d^{(1)},\alpha)$	$a_2 \cdot (\frac{1}{210} l^7 + \frac{1}{72} l^6 u)$	$a_2 \cdot \frac{17}{20160}$
(α,α)	$a_2 \cdot \left(\frac{17}{10080} l^8 + \frac{1}{210} l^7 u\right)$	$a_2 \cdot \frac{23}{90720}$

2.1 Proof of Corollary 1

Proof. We first evaluate $S_\alpha = I(\alpha, \alpha; T)$. We have $\alpha(n) \ll d(n)\log^2(n)$,

$$\sum_{n\le t} \frac{\alpha(n)^2}{n} = a_2 \cdot \frac{17}{20160}\log^8 t + O(\log^7 t)\ ,\ \textit{and}$$

$$\sum_{n\le t} \frac{\alpha(n)\alpha(pn)}{n} = a_2 \cdot \left(\frac{17}{10080}\log^8 t + \frac{1}{210}\log^7 t \log p\right) + O\left(\log^7 t + \frac{\log^8 t}{p}\right)$$

by Tables 1 and 3. Moreover, $\mathcal{A}(\alpha,\alpha) = a_2 \cdot \frac{23}{90720}$ by Lemma 10. Applying Lemma 5, we deduce

$$S_\alpha = \frac{TL^9}{2\pi}\left(a_2 \cdot \frac{17}{20160} - 2\mathcal{A}(\alpha,\alpha)\right) \sim \frac{61}{60480\pi^3} TL^9 = \frac{a_2 TL^9}{2\pi}\frac{61}{181440} \quad (94)$$

with an error term $O(TL^8 \log L)$. Next we consider

$$S_\beta = \sum_{0<\gamma<T} D_{\beta_\gamma}(\rho) D_{\beta_\gamma}(1-\rho) = \sum_{0<\gamma<T} \sum_{1\le m,n\le \frac{\gamma}{2\pi}} \frac{\beta_\gamma(m)\beta_\gamma(n)}{m^\rho n^{1-\rho}}\ . \quad (95)$$

Before evaluating S_β, we require some notation. For $N \in \mathbb{Z}_{\ge 0}$, $a(n)$ and $b(n)$ sequences, define

$$I_N(a,b;T) = \sum_{0<\gamma<T} \log^N\left(\frac{\gamma}{2\pi}\right) D_a(\rho) D_b(1-\rho)\ .$$

Notice that $I_0(a,b;T) = I(a,b;T)$ of (18). Observe that if $a(n)$ and $b(n)$ are real sequences then $I_N(b,a;T) = \overline{I_N(a,b;T)}$ by a consideration similar to (24). Applying the identity $\beta_t(m) = l^2 d(m) - 2l d^{(1)}(m) + \alpha(m)$ where $l = \log(\frac{t}{2\pi})$ we obtain

$$\begin{aligned}\beta_t(m)\beta_t(n) = {} & l^4 d(m)d(n) - 2l^3 d(m)d^{(1)}(n) + l^2 d(m)\alpha(n) \\ & - 2l^3 d^{(1)}(m)d(n) + 4l^2 d^{(1)}(m)d^{(1)}(n) - 2l d^{(1)}(m)\alpha(n) \\ & + l^2\alpha(m)d(n) - 2l\alpha(m)d^{(1)}(n) + \alpha(m)\alpha(n)\ .\end{aligned} \quad (96)$$

Inserting (96) in (95) we obtain

$$\begin{aligned}S_\beta = {} & I_4(d,d;T) + 4I_2(d^{(1)},d^{(1)};T) + I(\alpha,\alpha;T) \\ & - 4\mathrm{Re}(I_3(d,d^{(1)};T)) + 2\mathrm{Re}(I_2(d,\alpha;T)) - 4\mathrm{Re}(I_1(d^{(1)},\alpha;T))\ .\end{aligned} \quad (97)$$

Note that if $I(a,b;T) = c_1 T \log^M T + O(T\log^{M-1} T)$ then partial summation implies $I_N(a,b;T) = I(a,b;T)(\log^M T + O(\log^{M-1} T))$. In an analogous calculation to that of S_α, we derive by Lemma 5, Tables 1-3, and the partial summation identity the following:

$$I(d,d;T) \sim \frac{a_2 TL^9}{2\pi}\left(\frac{1}{24} - 2\cdot\frac{1}{60}\right) = \frac{a_2 TL^9}{2\pi}\cdot\frac{1}{120}\ , \tag{98}$$

$$I(d^{(1)},d^{(1)};T) \sim \frac{a_2 TL^9}{2\pi}\left(\frac{1}{144} - 2\cdot\frac{1}{420}\right) = \frac{a_2 TL^9}{2\pi}\cdot\frac{11}{5040}\ , \tag{99}$$

$$I_3(d,d^{(1)};T) \sim \frac{a_2 TL^9}{2\pi}\left(\frac{1}{60} - \frac{1}{144} - \frac{1}{180}\right) = \frac{a_2 TL^9}{2\pi}\cdot\frac{1}{240}\ , \tag{100}$$

$$I_2(d,\alpha;T) \sim \frac{a_2 TL^9}{2\pi}\left(\frac{1}{180} - \frac{1}{420} - \frac{1}{630}\right) = \frac{a_2 TL^9}{2\pi}\cdot\frac{1}{630}\ , \tag{101}$$

$$I_1(d^{(1)},\alpha;T) \sim \frac{a_2 TL^9}{2\pi}\left(\frac{1}{420} - \frac{17}{20160} - \frac{1}{1440}\right) = \frac{a_2 TL^9}{2\pi}\cdot\frac{17}{20160} \tag{102}$$

where each of these holds with an error term $O(TL^8\log L)$. By (97), (94) and (98) - (102) we have

$$S_\beta \sim \frac{a_2 TL^9}{2\pi}\left(\frac{1}{120} + 4\cdot\frac{11}{5040} + \frac{61}{181440} - 4\cdot\frac{1}{240} + 2\cdot\frac{1}{630} - 4\cdot\frac{17}{20160}\right)\ .$$

This simplifies to $S_\beta = \frac{97}{60480\pi^3} TL^9 + O(TL^8\log L)$.

2.2 Proof of Theorem 2

Proof. By Tables 1 and 2 we have

$$T_{d,d}(t) = a_2\cdot\frac{1}{24}\log^4 t + O(\log^3 t)\ ,\ T_{d,d;p}(t) = a_2\cdot\frac{1}{12}\log^4 t + O(\log^3 t)\ . \tag{103}$$

Therefore by Lemma 3

$$I(d,d;T,\delta) = \frac{T}{2\pi}\left(\frac{a_2}{24}L^5 - 2\mathrm{Re}\left(M\left(d,d;\frac{T}{2\pi},\delta\right)\right)\right) + O(TL^4\log L) \tag{104}$$

where $\delta = \frac{\lambda}{L}$. Since we have (103) an application of Lemma 4 yields

$$M\left(d,d;\frac{T}{2\pi},\delta\right) = L^5\sum_{k=0}^{\infty}\frac{(i\delta L)^k}{k!}\frac{a_2}{12}\frac{k!4!}{(5+k)!} + O(L^4)\ . \tag{105}$$

Thus (104) and (105) imply

$$I(d,d;T,\delta) = \frac{a_2 TL^5}{2\pi}\left(\frac{1}{5!} - 4\sum_{j\geq 1}\frac{(-1)^j\lambda^{2j}}{(5+2j)!}\right) + O(TL^4\log L) \qquad (106)$$

and we are finished.

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